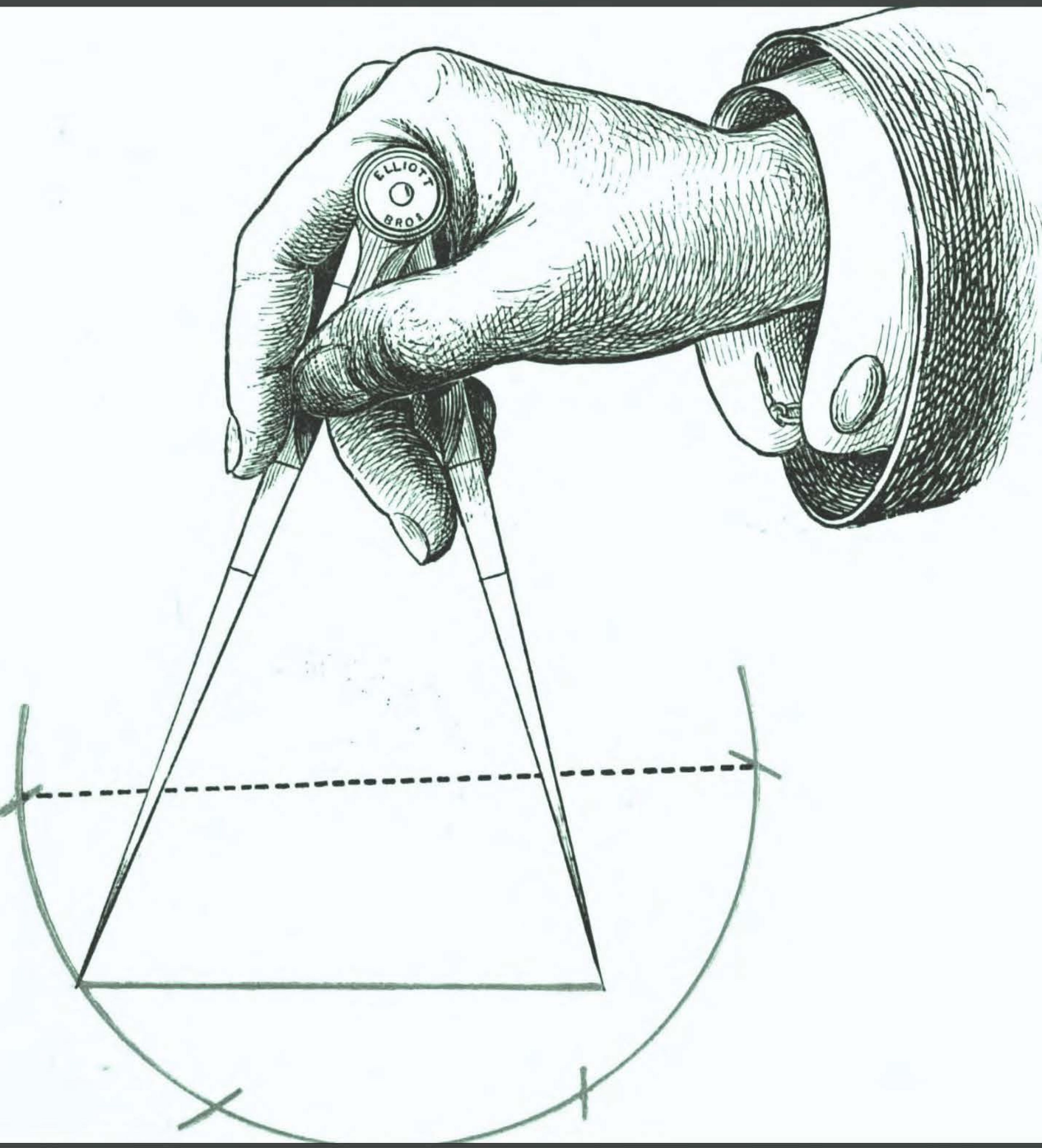
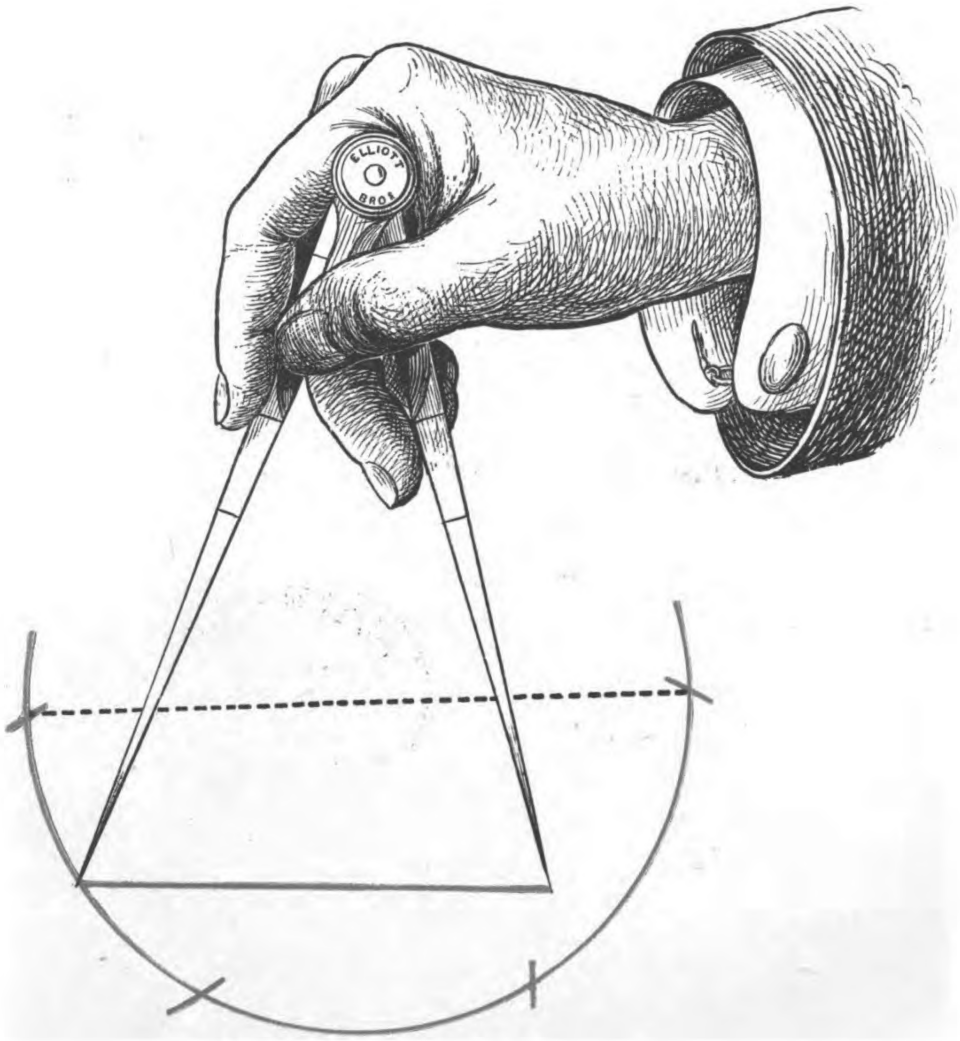


The geometry of compasses; or, Problems resolved by the mere ...

Oliver Byrne

THE GEOMETRY OF COMPASSES.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THE

GEOMETRY OF COMPASSES

OR PROBLEMS RESOLVED BY THE MERE DESCRIPTION OF CIRCLES,

AND THE USE OF COLOURED DIAGRAMS AND SYMBOLS.

BY OLIVER BYRNE,

AUTHOR OF "THE ELEMENTS OF EUCLID BY COLOURS,"
"THE YOUNG GEOMETRICIAN," &C.

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PREFACE.

THE problems resolved in this work, by the description of circles only, are the most important problems in practical geometry. How to describe a circle with a pair of compasses is all the previous knowledge required by a geometrical student.

In my work entitled "The Young Geometrician," and in my Euclid, I employed coloured diagrams, signs, and symbols, to carry on demonstrations, which rendered the reasoning more precise, and the attainment more expeditious; in the present work, by the aid of coloured diagrams, I also effect my object with ease, precision, and dispatch.

All the coloured traces of circular arcs are described or traced with radii, or straight distances, traced in corresponding colours; as the student proceeds, he is often reminded of this conventional arrangement. The young student should clearly comprehend how lines, which possess length only,

without breadth, are represented or indicated by coloured traces; straight lines and circular arcs, being without breadth, are supposed to divide evenly their coloured traces into two equal parts in the direction of their lengths.

It is also necessary to observe that the demonstrations of the problems are designed for advanced geometrical students.

OLIVER BYRNE.

PROBLEMS.

PROBLEM I.

LET A and B be two given points; it is required to find, by means of compasses only, a series of equidistant points in the straight line A B produced.

The circular arcs traced in *black* are described with the radius marked *black;* the straight line traced in *red* is the radius of the circular arcs marked *red;* the *blue* straight line is the radius of the *blue* arcs; and so on.

The *black*, *red*, *blue*, &c., straight lines are equal and will lie in the same straight line. Mathematical lines and arcs, being without breadth and thus indicated, are supposed to divide evenly their coloured traces into two equal parts, in the direction of their lengths.

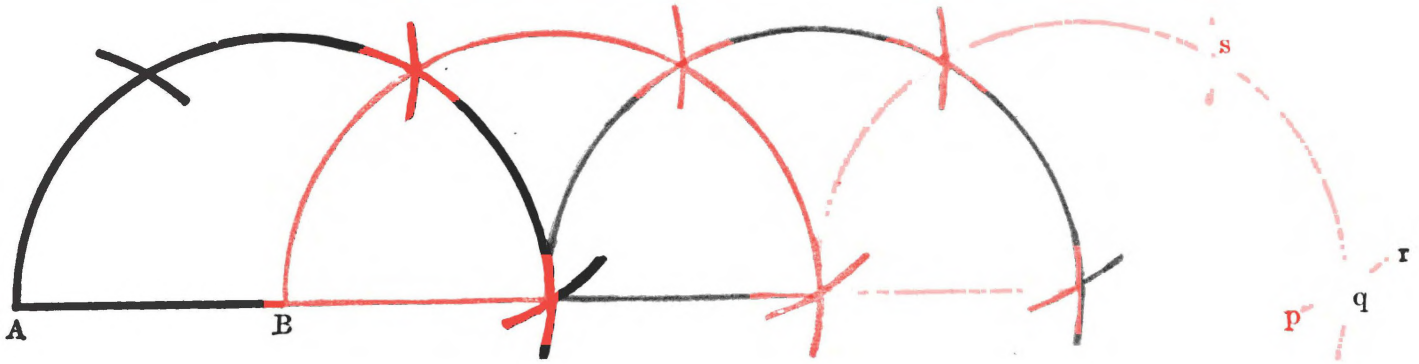

Fig. 1.

DEMONSTRATION OF PROBLEM I.

FROM the construction, it is evident that three equilateral triangles are formed about the point B, and hence (Euclid, I. 32) the equal angles A B *m*, *m* B*n*, *n* B C, taken together, make two right angles; therefore, A B, B C, are in the same straight line. Similar reasoning applies to the successive points C, D, E, &c. Fig. 2.

Observe. The radius being given; to find the centre from which any given small arc as *p q r*, Fig. 1, has been described. The given radius being applied, with *q* as a centre, the point *s*, the centre of the arc *p q r*, becomes known.

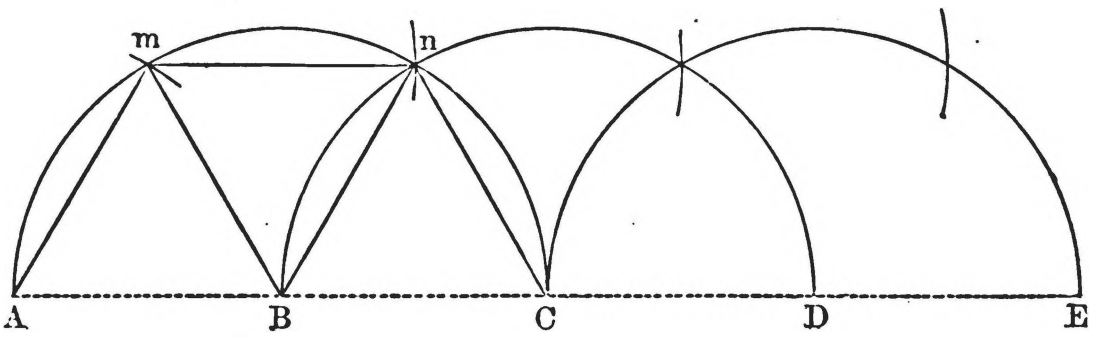

Fig. 2.

PROBLEM II.

GIVEN the points A and B; to find a third point C, such that the straight line connecting C with B shall be at right angles to A B, Fig. 3.

The circle and arcs indicated by the *red* traces are all described with any convenient radius, such as the one indicated by the straight *red* trace. Then the straight line pointed out by the *blue* trace is perpendicular to the straight line connecting A with B.

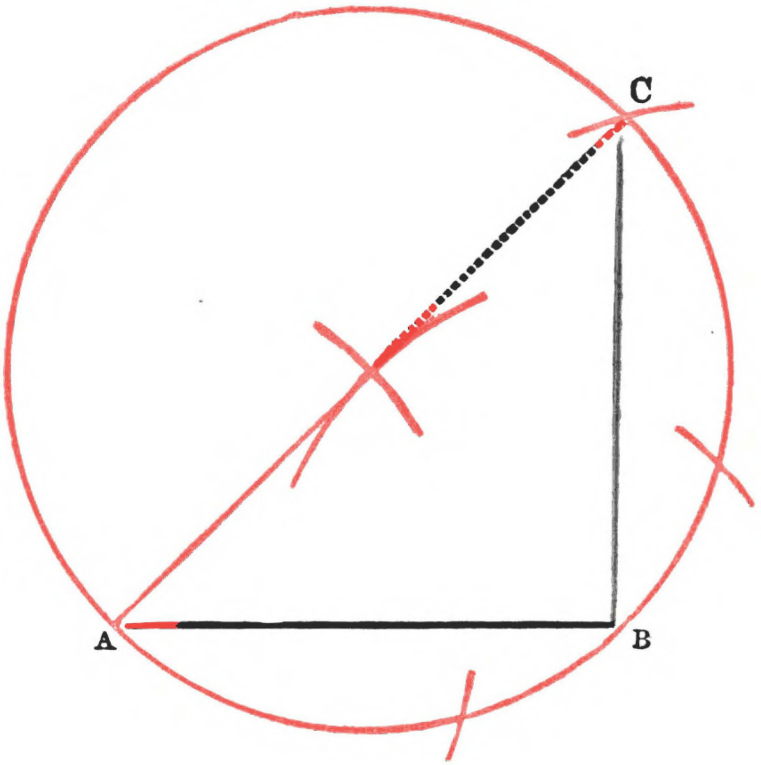

Fig. 3.

DEMONSTRATION OF PROBLEM II.

FROM A and B, with any convenient distance, describe two arcs intersecting in O, Fig. 4, from which, with the same radius, describe a circle passing through the points A and B; insert the same radius, A O, three times from A to C; then C B is perpendicular to A B.

It is evident, from the last proposition (1), that the arc A B C is a semicircle, and consequently (Euclid, III. 31) the angle A B C contained in it is a right angle. The line A B may be taken as the radius to operate with.

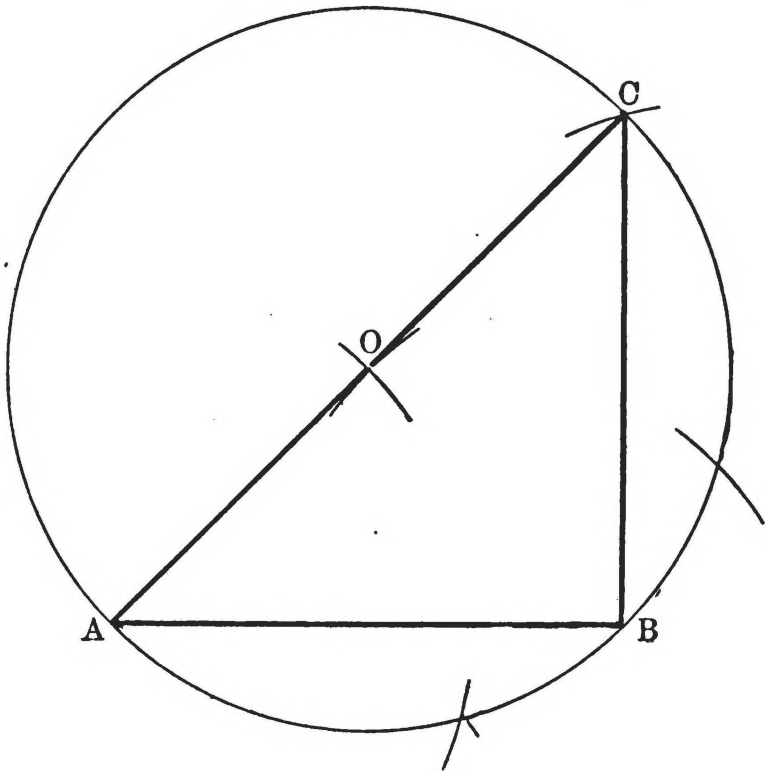

Fig. 4.

PROBLEM III.

TO find the direction of a perpendicular, let fall from a given point C, upon a straight line which connects two given points, A, B.

Let C be a point, from which a perpendicular has to be let fall upon the straight line joining A and B; Fig. 5.

With A as a centre, describe the arc traced in *red* with the straight line traced in the same colour; describe the *blue* arc with the *blue* line; then the *black* dotted line is perpendicular to the *black* line that joins A and B.

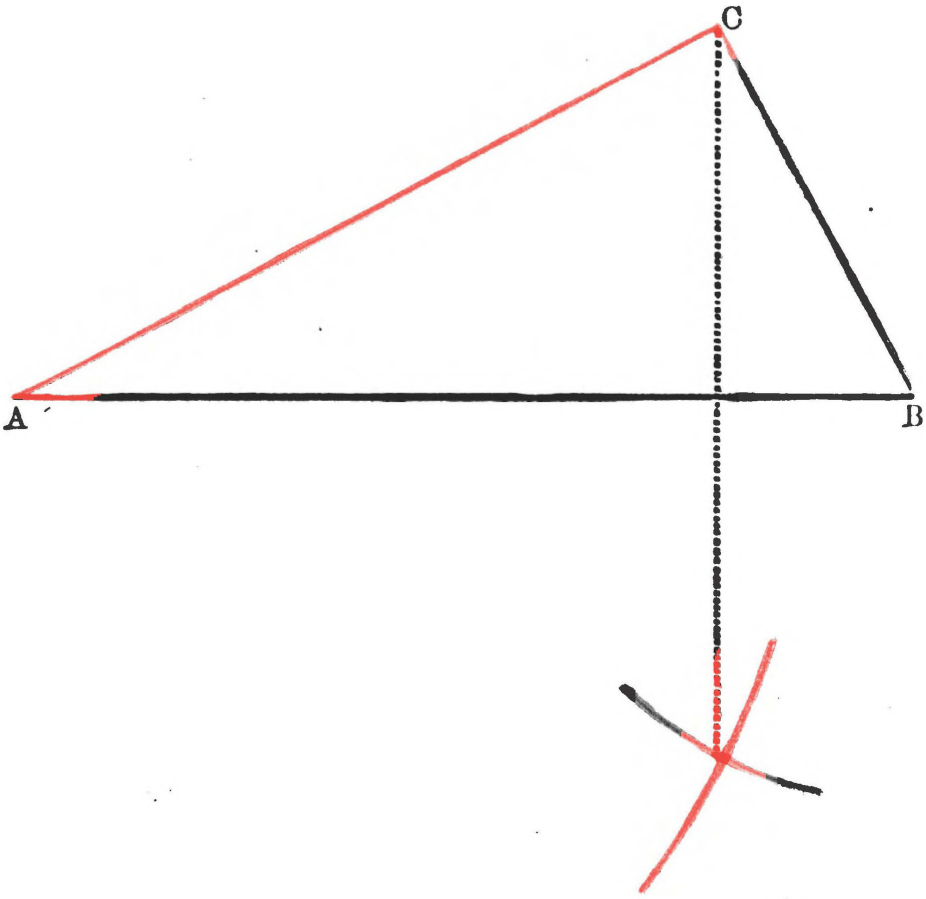

Fig. 5.

DEMONSTRATION OF PROBLEM III.

FROM A as a centre, with the distance A C, describe an arc, and from B as a centre, with the distance B C, describe another arc, intersecting the former in the point E; then, C E is perpendicular to A B. For the triangles A C E and B C E are isosceles, and therefore their vertices must lie in a straight line A B which bisects the common base C E at right angles (Euclid, I. 12).

Let $C B_1 = C B = b$; $A C = a$; $A B_1 = c$; and $B_1 D = x = D B$; Fig. 6.

Then (Euclid, II. 13), $a^2 = b^2 + c^2 + 2\ (c \times x)$; therefore, $a^2 - b^2 = c\,(c + 2x) = A B_1 \times (A B)$; for, $c^2 + 2\ (c \times x) = c\,(c \times 2x)$.

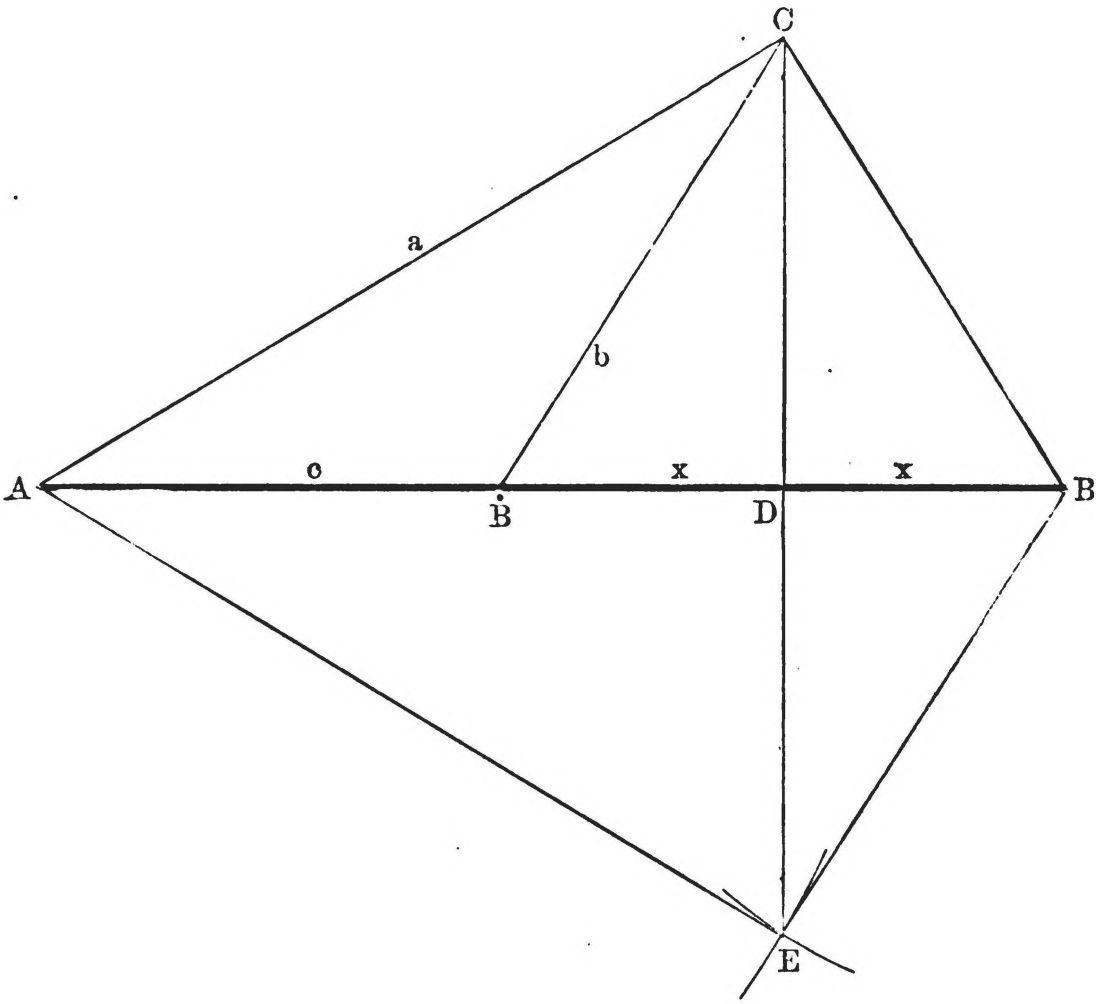

Fig. 6.

PROBLEM IV.

LET A and B be two given points; it is required to find the middle point C, by the mere description of circles.

With the radius A B describe the arcs marked *red;* as in the foregoing figures, the arc traced in *blue* is described with the radius indicated by the *blue* straight line; the *black* arcs are traced by the radius indicated by the *black* line. Then, the given straight line A B is divided into two equal parts in the point C, by the intersection of the *black* arcs.

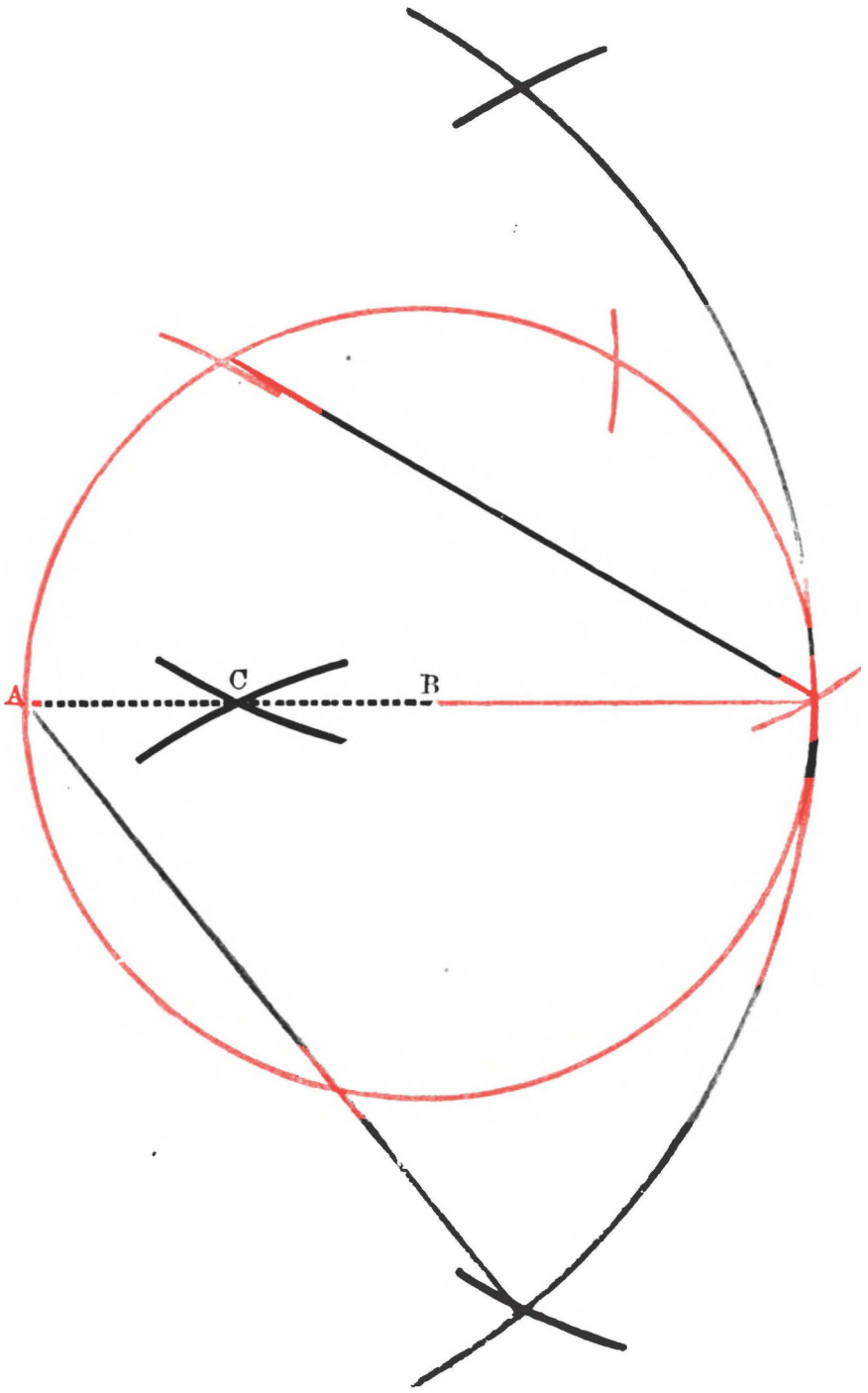

Fig. 7.

DEMONSTRATION OF PROBLEM IV.

FROM B as a centre, with radius B A, describe a semicircle, by inserting A B successively from A to K, from K to D, and D to E; from A as centre, with distance A E, describe a portion of a circle F E G, in which, and from E, lay off the chords E F, E G, equal to E K; and from the points F and G, with the same radius E K, describe arcs intersecting in C. The line A B is bisected by the point C. By the first problem, the points A, B, and E are in a straight line. On account of the isosceles triangles, C is in the same straight line.

Again, from the last problem, since E F C is an isosceles triangle, $A F^2 - F C^2 = E A \times A C$, that is, $A E^2 - E K^2 = A B^2 = E A \times A C$. Consequently, since E A is double A B; A B must be double A C.

If the points A and K, K and E, be joined; A K E will be a right-angled triangle, in which A K = A B.

Then $A K^2 = A B^2 = A E^2 - K E^2$, (Euclid I. 47).

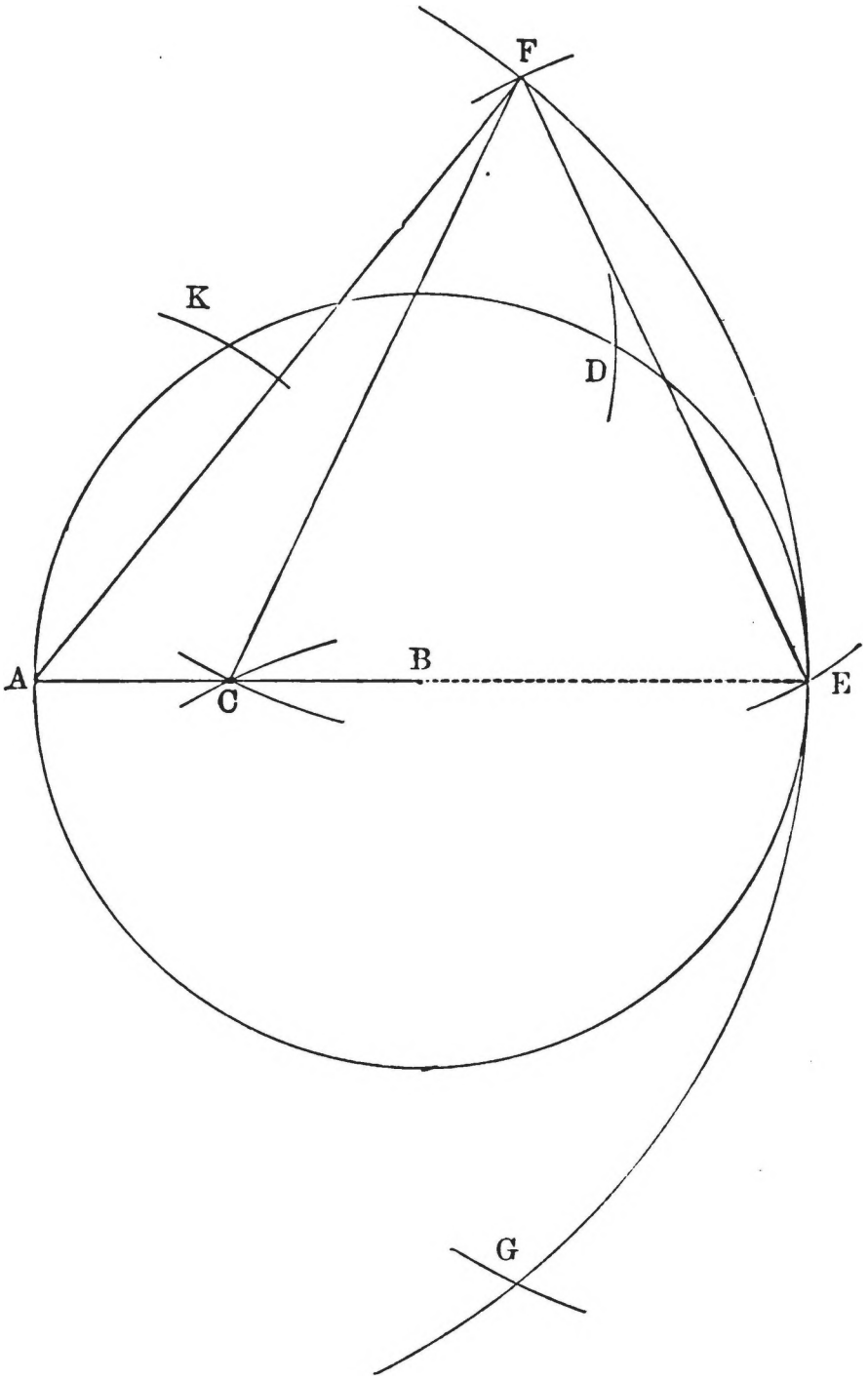

Fig. 8.

PROBLEM V.

LET it be required to bisect the arc A C B of a circle whose centre is O, Fig. 9.

The circular arcs traced in *red* are described with the radius traced in *red;* then the arcs traced in *black* are drawn by the radius indicated by the same colour. The general system of operating with colours being continued, the *yellow* arcs are described with the *yellow* radius; and the *blue* arc with the *blue* radius. The *blue* arc divides the given arc A C B into two equal parts, A C, C B, Fig. 9.

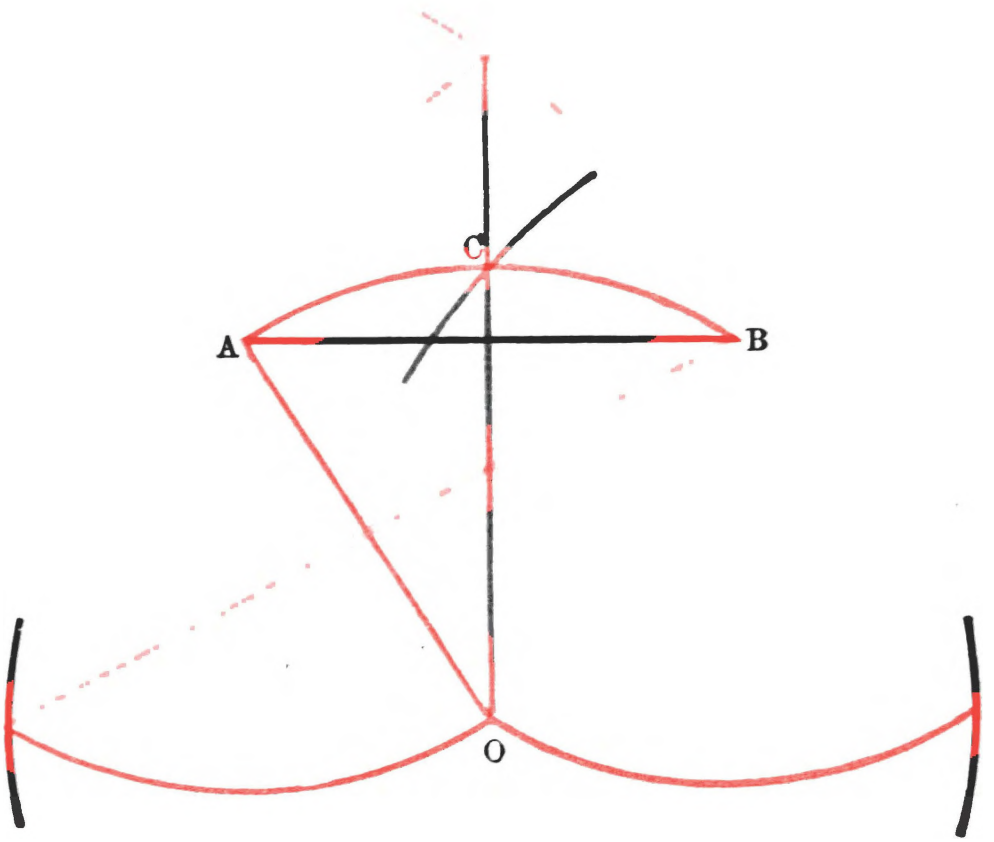

Fig. 9.

DEMONSTRATION OF PROBLEM V.

IT is evident A B O D and A B E O are rhomboids, and since D O and O E are parallel to A B; D O, O E are in the same straight line. The triangles D O F and F O E having their corresponding sides equal, the angle D O F is a right angle; and (Euclid, I. 47) $DF^2 = DO^2 + OF^2$; Fig. 10.

In the rhomboid A B O D; $DB^2 + AO^2 = 2DO^2 + 2OB^2$; or $DB^2 = 2DO^2 + OB^2$; and since D B = D F, $2DO^2 + OB^2 = DO^2 + OF^2$, therefore, $DO^2 + OB^2 = OF^2$ or $DO^2 + OC^2 = DC^2$, and therefore D O C is a right angle. Because O C is perpendicular to D O, it is likewise perpendicular to A B, and the triangles O A P and O B P are equal, and the angle A O C equal to B O C; consequently the arc A C = the arc B C (Euclid, III. 26).

It has to be borne in mind that DF^2 signifies the square or second power of the line D F; and that $2DO^2$ represents twice the square of, or on, the straight line D O.

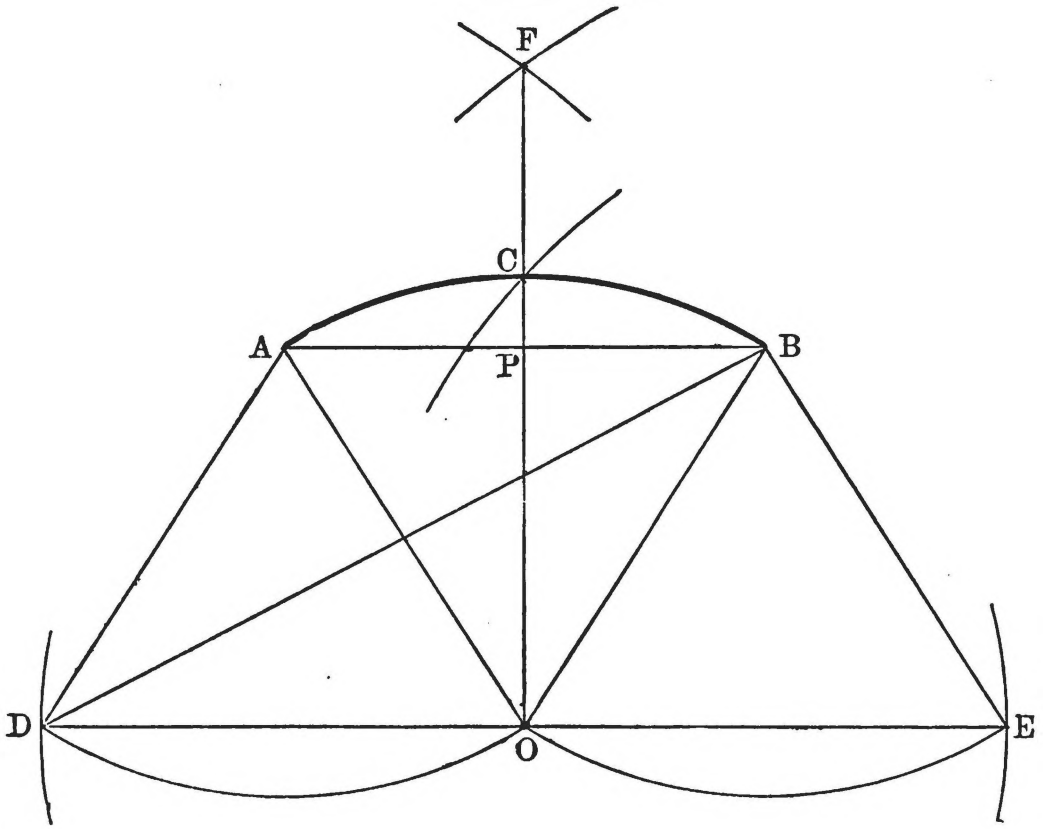

Fig. 10.

PROBLEM VI.

LET the points A and B terminate the side of a square, which it is required to trace by compasses only.

In Fig. 11, the semicircle and the other circular arcs traced in *red* are described with the given distance A B traced in *red.* Then the *blue* arcs are described with the *blue* straight line taken as radius, and the *yellow* arc with the *yellow* line. The intersection of the *red* arcs at C completes the square C B, Fig. 11.

The student must not forget that mathematical lines, being without breadth, and thus indicated by coloured traces, are supposed to divide evenly their coloured representatives into two equal circular arcs, or into two equal straight lines as the case may be.

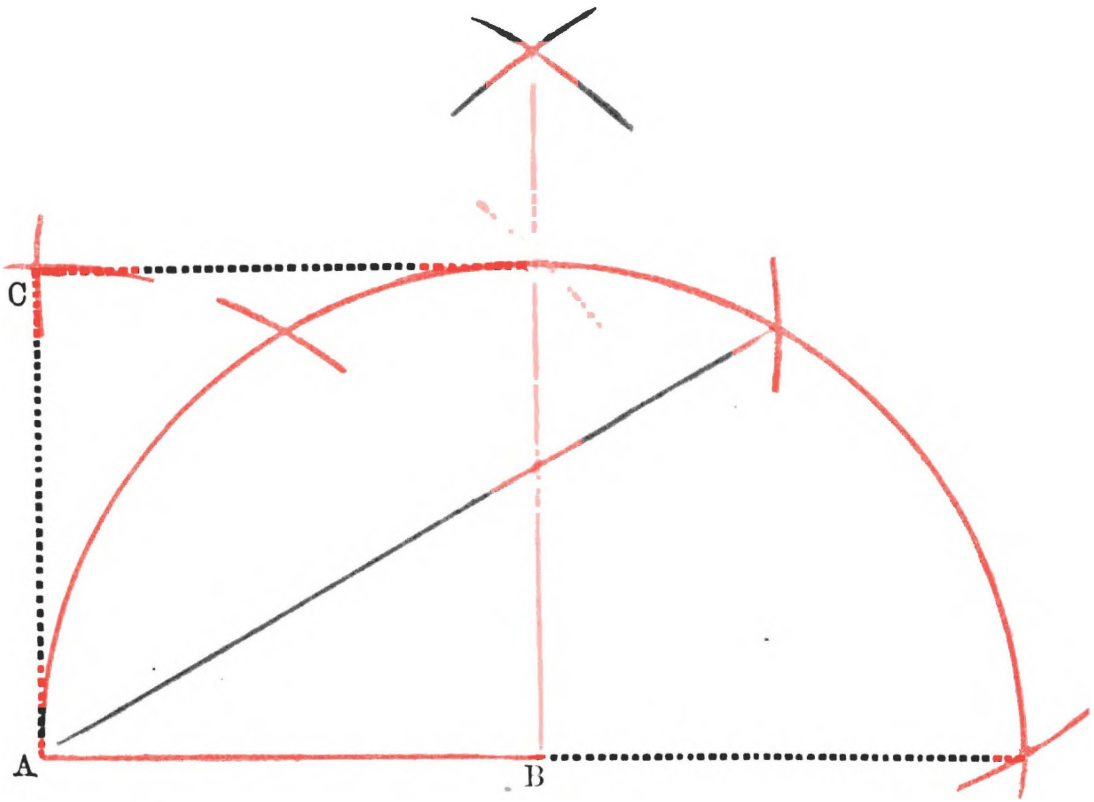

Fig. 11.

DEMONSTRATION OF PROBLEM VI.

FROM B as a centre describe the semicircle A D E F, Fig. 12; from A and F, with the distance A E, describe arcs intersecting in Q; from A with the distance B Q, cut the circumference in P, and from A and P, with the radius A B, describe arcs intersecting in C; the points C and P are the corners of the required square. For the angle A B P is a right angle, and the distances A B, A C, C P, and P B are by construction all equal.

$A\,E^2 + E\,F^2 = A\,F^2$, or, $A\,E^2 + A\,B^2 = 4\,A\,B^2$, that is, $A\,Q^2 = A\,E^2 = 3\,A\,B^2$, but, $A\,Q^2 = A\,B^2 + B\,Q^2$; therefore, $B\,Q^2 = A\,P^2 = 2\,A\,B^2$; hence, A P = B Q, is the diagonal of the required square.

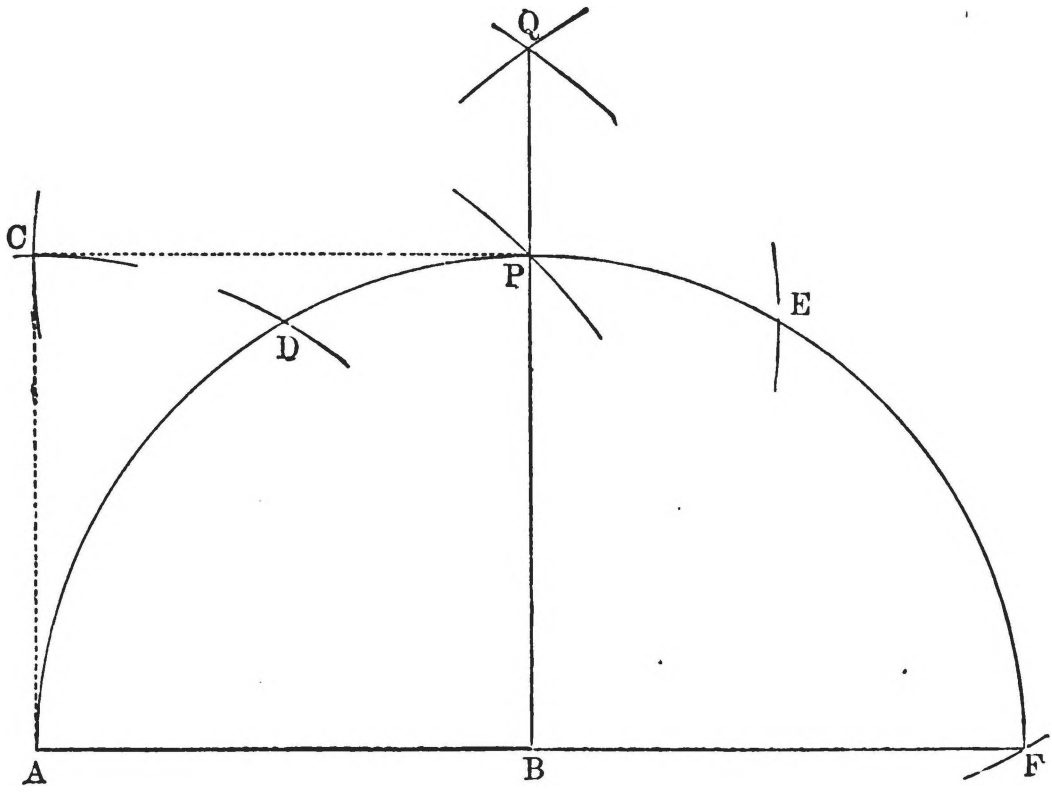

Fig. 12.

PROBLEM VII.

ON a given diagonal A B, to describe a square, by the mere description of circles.

Describe the semicircle and arcs indicated in *red* by the radius A B, Fig. 13, indicated by the same colour; in all cases a semicircle is marked off by inflecting the radius three times in succession. With the double chord traced in *black* describe the *black* arcs; with the centre C, and the *yellow* trace as radius, describe the arc marked *yellow;* and lastly from A and B with the *blue* line describe the *blue* arcs, and the four corners of the required square are determined.

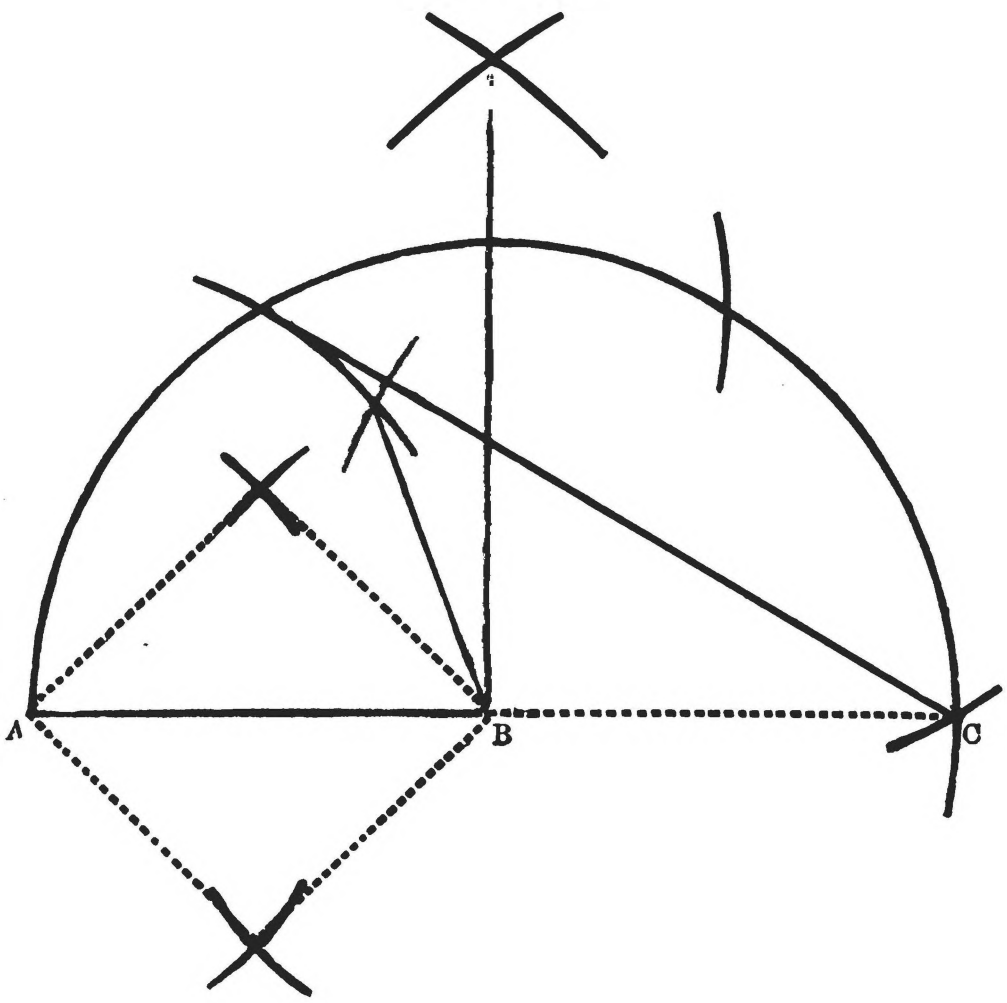

Fig. 13.

DEMONSTRATION OF PROBLEM VII.

FROM B as a centre describe the semi-circle A D A C, Fig. 14, from A and C with the double chord C D describe arcs intersecting in F; from C with the distance B F describe an arc and cut it from A with the radius A D in G; and from B and A with the distance B G describe arcs intersecting in H and K; then A H B K is the square required.

For in the triangle A G C, the straight line G B bisects the base A C, and therefore, $AG^2 + CG^2 = 2AB^2 + 2BG^2$; but, $CG^2 = BF^2 = 2AB^2$; therefore, $AG^2 = AB^2 = 2BG^2$; and A H B is a right angle; and the sides A H, H B, B K and K A being all equal, A H B K is a square.

Put $AB = EC = a$; $AC = 2a$; and since A E C is a right-angled triangle, the square of $AE = CD^2 = 4a^2 - a^2 = 3a^2$. Whence, the square of $AF = 3a^2 = CF^2$. And since A B F is a right-angled triangle, $AF^2 - AB^2 = BF^2$, that is, $3a^2 - a^2 = 2a^2$, and the square of B F = twice the square of A B.

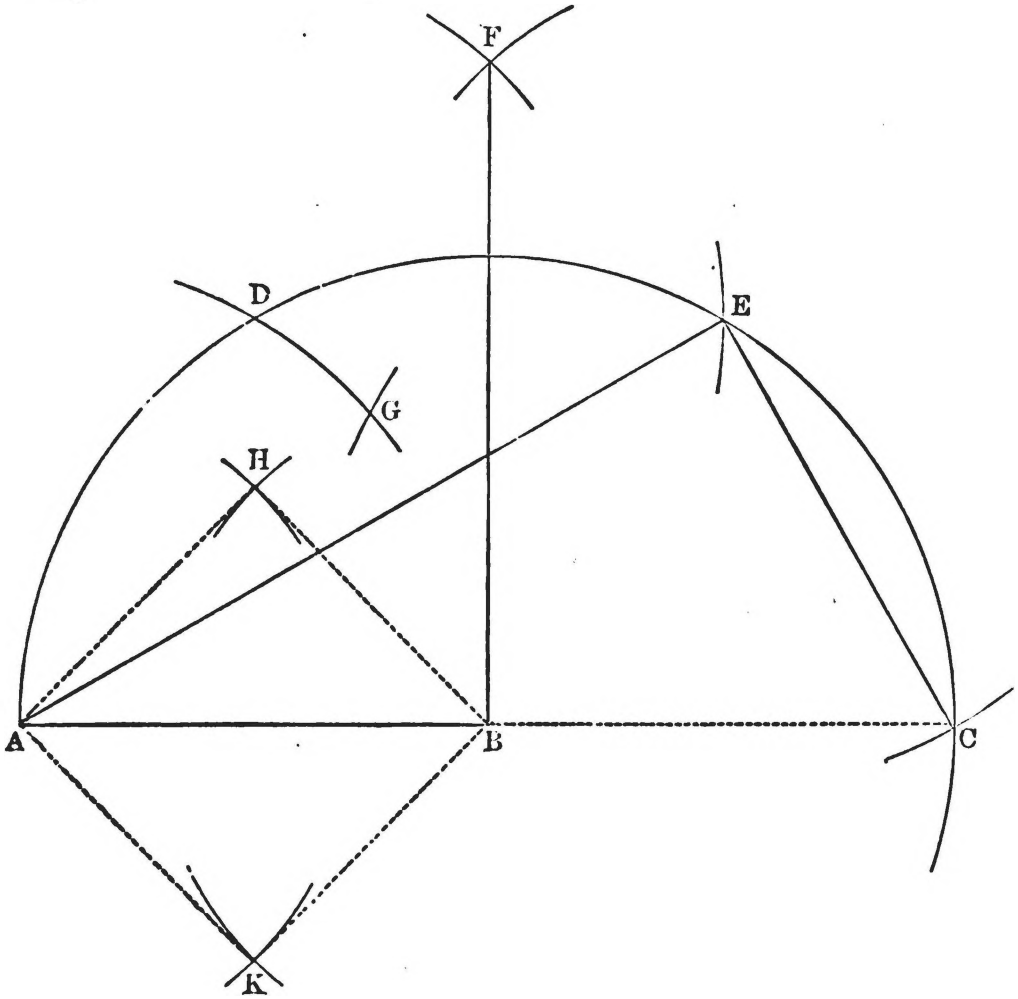

Fig. 14.

PROBLEM VIII.

GIVEN the side A B of a regular pentagon, to find the trace of the figure, by the use of compasses only.

Mere inspection shows, as in former constructions, how the dotted trace, Fig. 15, of the regular pentagon is determined.

The circular arcs marked *black* are described with the radius marked *black*. The straight line traced in *yellow* is the radius of the circular arcs marked *yellow*. The *blue* line is the radius of the *blue* arcs; and the *red* line is the radius of the *red* arcs.

Mathematical lines and arcs, being without breadth and thus indicated, are supposed to divide evenly their coloured traces into two equal parts, in the direction of their lengths. This observation is continually repeated, so that the mathematical lines indicated may not be lost sight of in operating with their coloured traces.

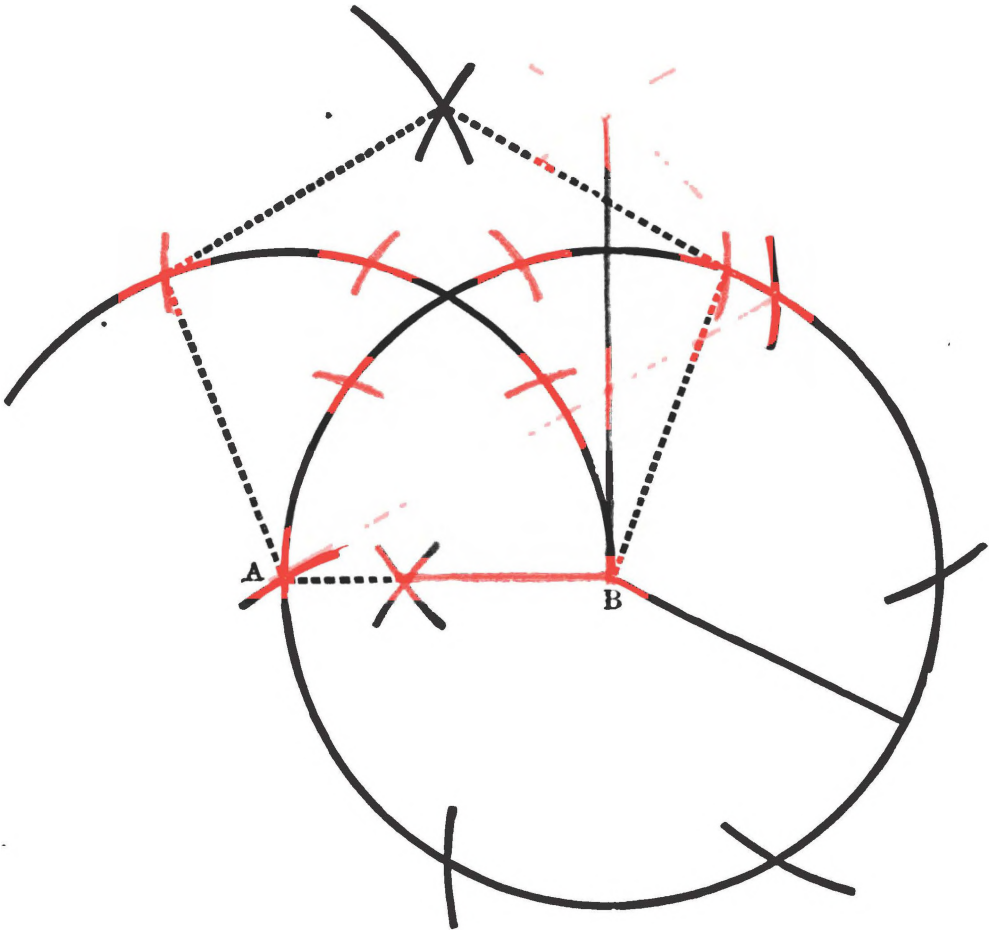

Fig. 15.

DEMONSTRATION OF PROBLEM VIII.

FROM B describe through A the circle A D E C F, in which the radius is laid off four times; from A and C with the double chord A E describe arcs intersecting in G; from E and F, with the distance B G, describe arcs intersecting in H; from A, with the radius A B, describe a portion of a circle, lay off B H thrice from B to L; and thrice from A to O. Then, from L and O, with the radius A B, describe arcs intersecting in P. The points A, L, P, O, B, are the angular points of the required regular pentagon, Fig. 16.

From Problem IX., it is evident that B H is the greater segment of the distance A B divided in extreme mean ratio. Consequently (Euclid, book IV., 10; and II., 11), the isosceles triangles B A I, I A K, K A L, A B M, M B N, and N B O, have each of the angles at the base double their vertical angle. Therefore the angles B A L and A B O are each of them six-fifths of a right angle, and hence the points L and O are corners of the pentagon. P is evidently the vertex of the pentagon, for the sides L P and O P are each equal to A B. Let x represent the angle B A I, then A I B $= 2x$; and A B I $= 2x$; $x + 2x + 2x = 180°$, or $5x = 180$, $\therefore x = 36°$. $36 \times 3 = 108°$ $=$ B A L $= 90 \times \frac{6}{5}$. (Euclid, I. 32, corollary.)

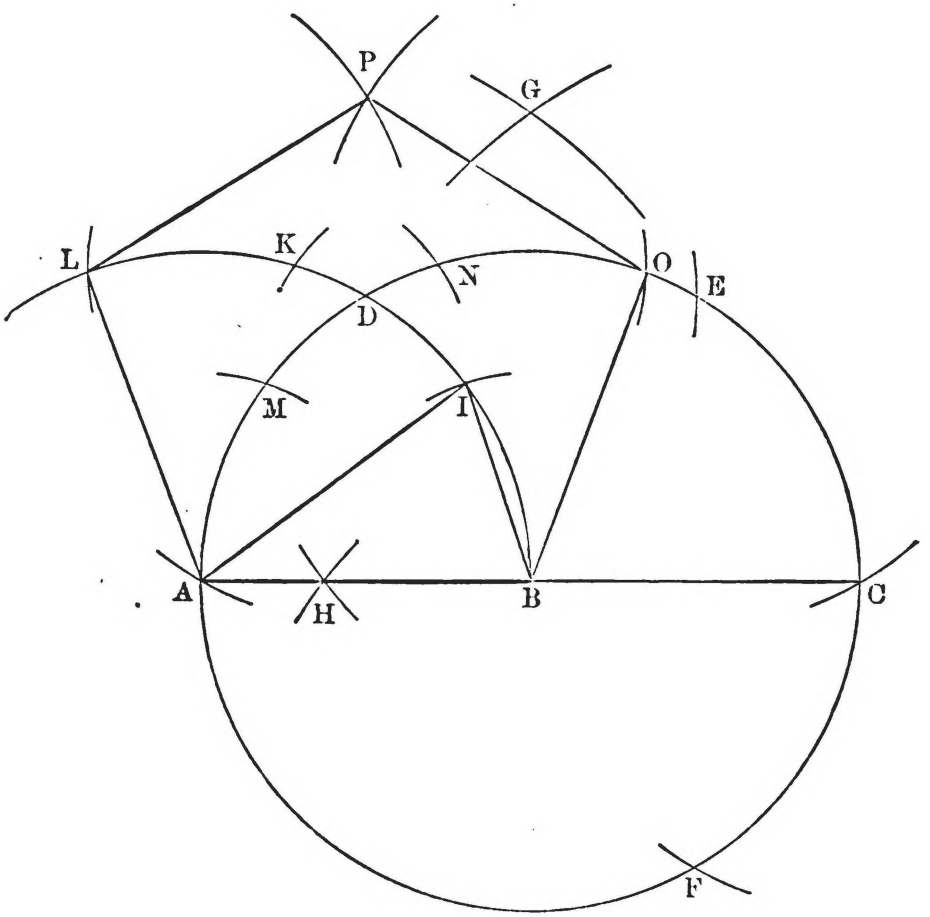

Fig. 16.

PROBLEM IX.

TO divide a given distance A B in an extreme and mean ratio in the point C, or to find a point C, so that $B A \times C A = B C^2$.

Describe the circle and the circular arcs traced in *red* with the corresponding radius traced in *red;* and, according to our conventional arrangement, describe the *blue* arcs with the *blue* radius; lastly, describe the arcs marked *black* with the radius indicated by the *black* trace. Then the *black* arcs at C divide the *red* distance, or line A B, into two such parts as make $B A \times A C = C B^2$; Fig. 17.

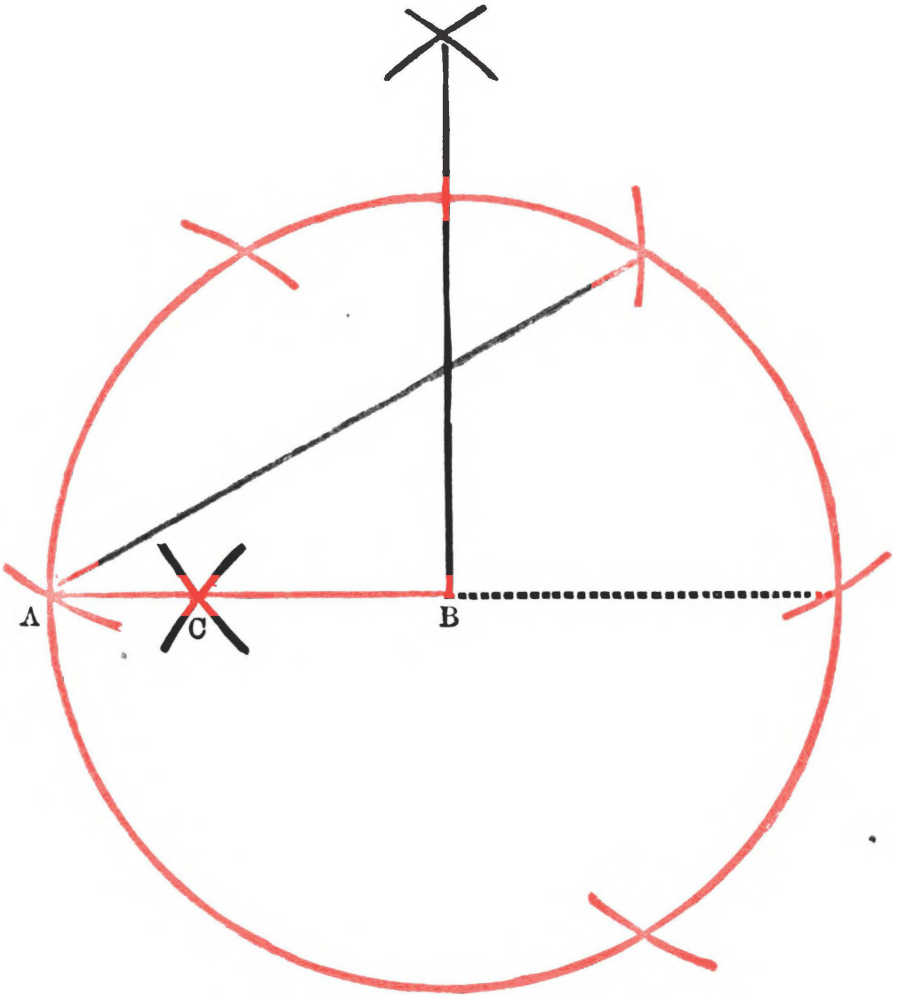

Fig. 17.

DEMONSTRATION OF PROBLEM IX.

FROM B describe a circle with the radius B A, Fig. 18; insert the radius B A successively from A to D, E, K, and F; from the extremities of the diameter A K, and with the double chord A E, describe two arcs intersecting in G; and from the points E and F, with the distance B G, describe two other arcs intersecting in C; then, $BA \times AC = CB^2$. For, from the nature of isosceles triangles, it is evident that the point C lies in the straight line AB. AEK is a right-angled triangle, $AB = BE = EK = KB$. And because the triangles AGB, KGB, have their sides respectively equal, the angle ABG is a right angle, and therefore $AG^2 = AB^2 + BG^2$; $AG = AE$, and $AE^2 = 3AB^2$; therefore, $3AB^2 = AB^2 + BG^2$, or $BG^2 = 2AB^2$. But since $BE = EK$, it follows that $CE^2 - EB^2 = KC \times CB$; now $CE^2 - EB^2 = BG^2 - BE^2 = AB^2$; therefore, $AB^2 = KC \times CB = BK^2$. Hence, CK is cut in an extreme and mean ratio by the point B. Let $BK = a = BA$; $BC = b$, then $AC = a - b$; it has been shown that $(a + b) \times b = a^2$, or $ab + b^2 = a^2$; hence, $b^2 = a^2 - ab = a(a - b) \therefore CB^2 = AB \times AC$.

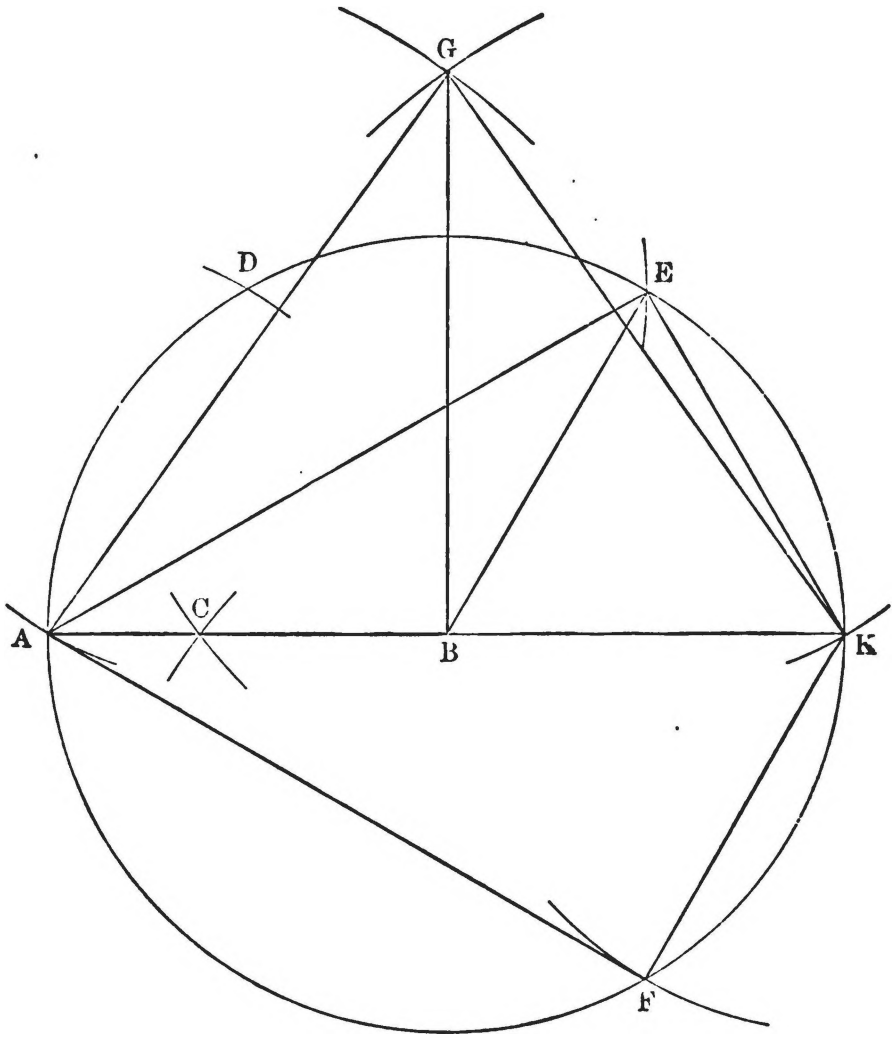

Fig. 18.

PROBLEM X.

THE centre of a circle being lost, it is required to find it, by the description of circles only.

To find the centre of the circle whose circumference is traced in *yellow;* at random, with any chord as the one traced in *red*, describe the semicircle and the other circular arcs indicated by the *same colour;* then with the radius traced in *blue* describe the circular arcs traced in *blue;* and lastly the point, where the *black* arcs described with the *black* chord as radius intersect, is the centre of the *yellow* circle; Fig. 19.

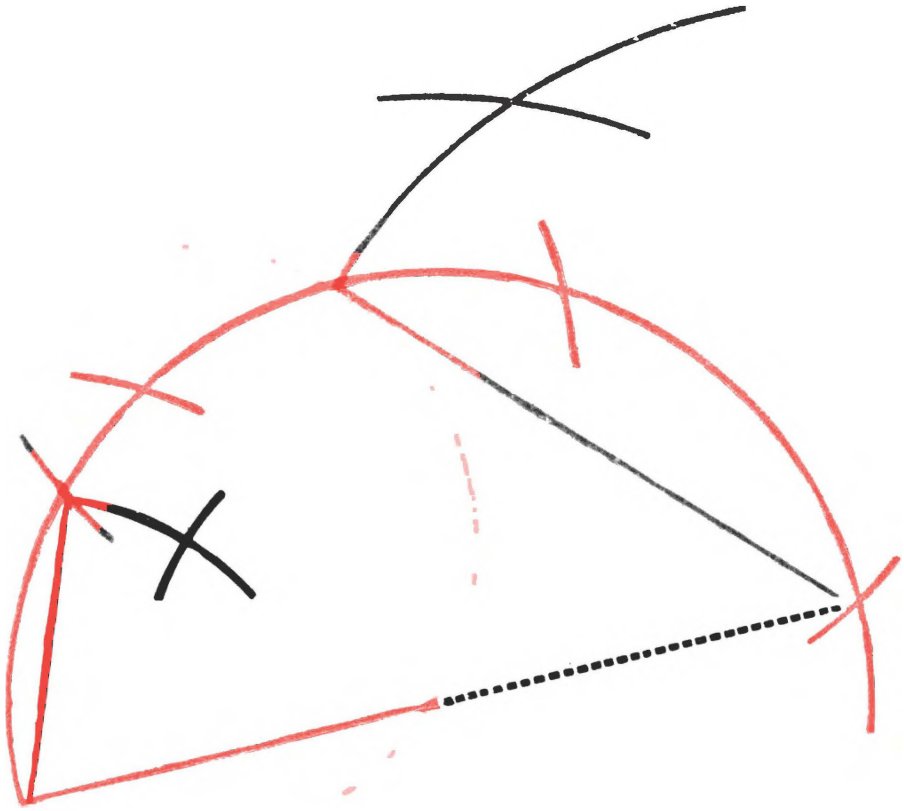

Fig. 19.

DEMONSTRATION OF PROBLEM X.

STRAIGHT lines being drawn, Fig. 19 assumes the construction shown in Fig. 20.

The isosceles triangles F E B, B E C, are evidently equal; and it is easily shown that each of these triangles is similar to the isosceles triangle A B F; for the angle F B C is equal to twice the angle F B E, and also equal to twice the angle F A B. Therefore the triangle F A B, being isosceles, is similar to the isosceles triangle F E B, since the exterior angle F B C is equal to the sum of the interior and opposite angles B F A, B A F; (Euclid, I. 32).

∴ B E : B F :: B A : A F, or, C D : D B :: B A : A G; consequently, the triangles A G B, and D B C, being isosceles, are also similar, and the angle B C D is equal to G B A; B G is therefore parallel to C D, and hence the angles B D C and G B D are equal. It is now evident that the triangles B G D and B A G are equal, for A B = B D and B G common, and the angle G B D = B D C = D C B = G B A. Consequently, the side A G is equal to the side G D. Hence the point G being equidistant from three points, A, D and B, in the circumference, G is the centre of the circle A B D; (Euclid, III. 9).

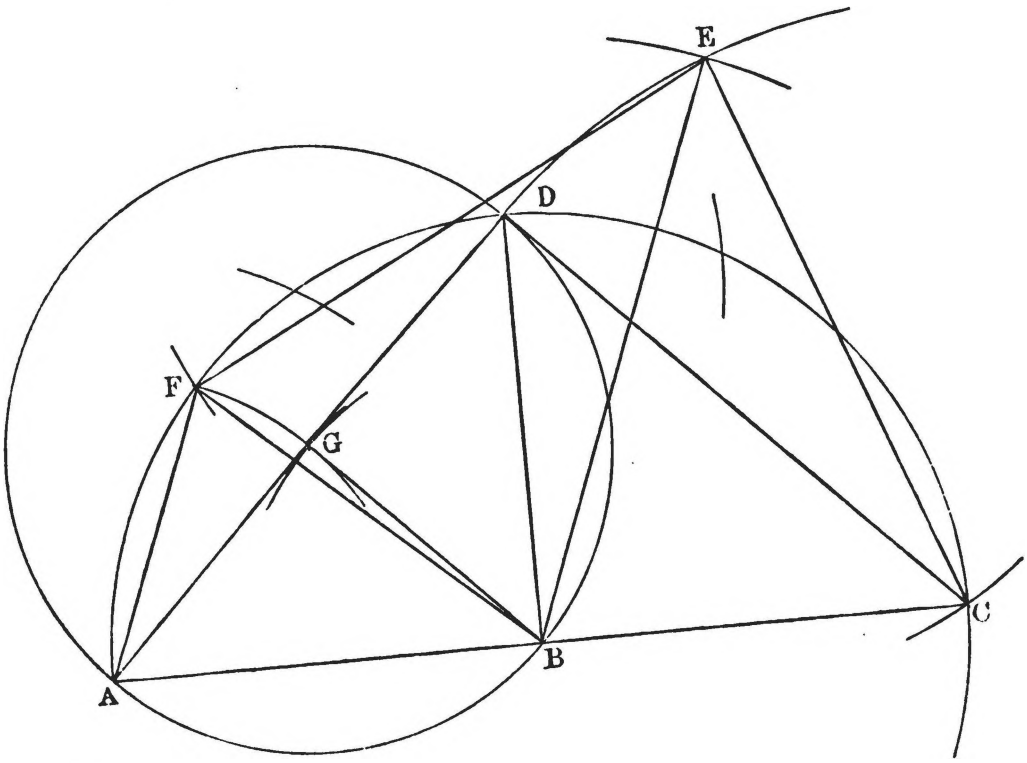

Fig. 20.

PROBLEM XI.

LET the side of a regular octagon terminate in the points A and B; it is required to find the remaining corners of the figure; Fig. 21.

The two semicircles and all the arcs traced in *red* are described with the given radius A B, Fig. 21, marked *red;* the *blue* arcs are described with the double chord traced in *blue;* and, lastly, describe the arcs exhibited by their *black* traces with the radius traced in *black.* The dotted straight lines connect the corners of the required regular octagon.

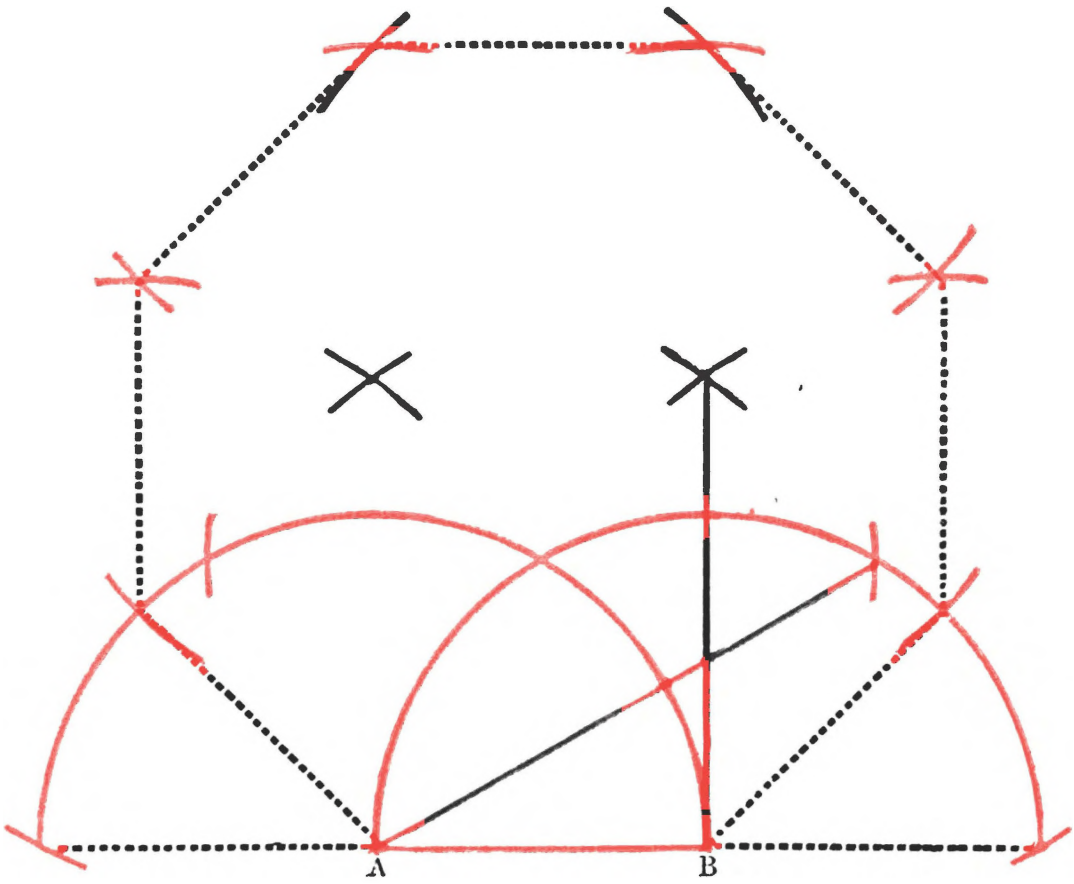

Fig. 21.

DEMONSTRATION OF PROBLEM XI.

N A B describe the two semicircles A E F C and B E G D; with the double chord A F, and from A, C and B, D describe arcs intersecting in H and I; from these points, with the radius A B, cut the semicircles in K and L; on H I describe the square H M N I, by making the diagonals H N, I M equal to B H, and the sides equal to A B; complete the rhombus M O K H, and the rhombus I N P L; the points A, B, K, O, M, N, P and L, are the several corners of the regular octagon; Fig. 22.

For it is evident that B H, A I are both perpendicular to A B, and B K H, A L I are right-angled isosceles triangles: H I is therefore parallel to B A, and H M N I, consisting of triangles equal to B K H, is a square; whence all the sides A B, B K, K O, O M, M N, N P, P L and L A of the octagon are equal. But they likewise contain equal angles; for A B K, composed of A B H and H B K, is equal to three half right angles, and B K O, by reason of the parallels B H and K O, being the supplement of H B K, is also equal to three half right angles. In a similar manner the other angles of the figure may be proved to be equal.

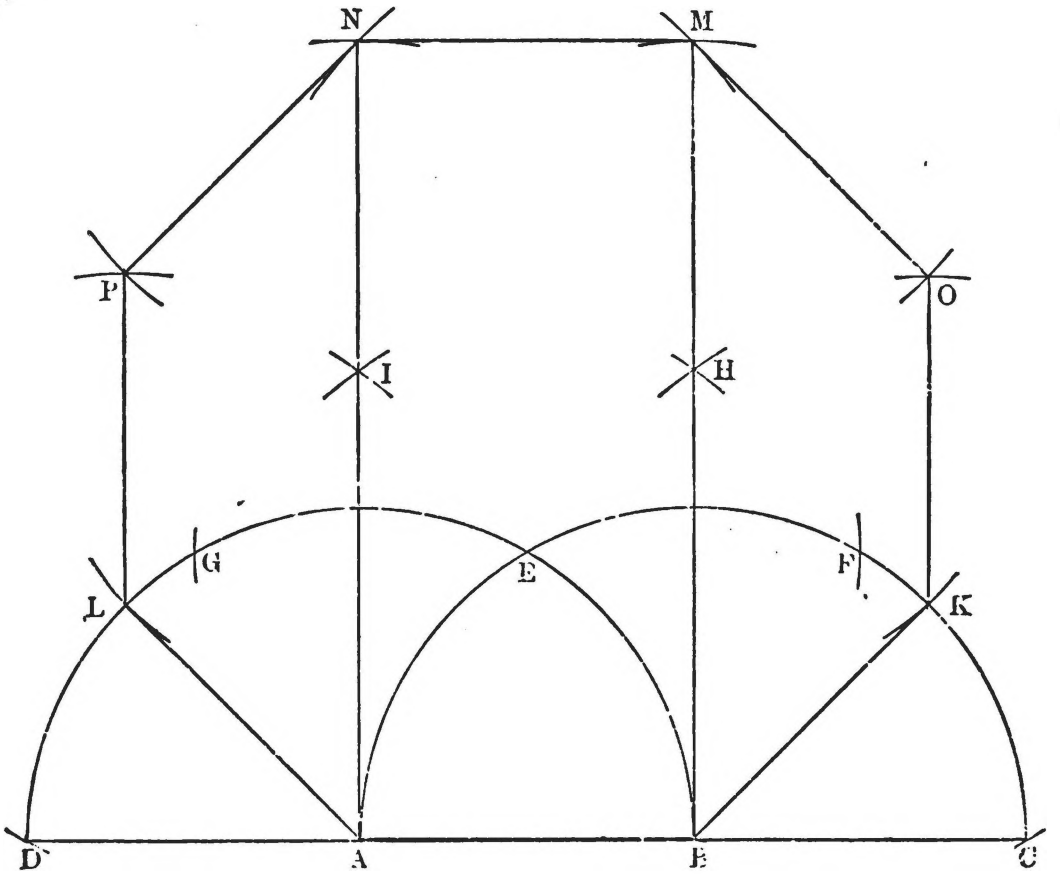

Fig. 22.

PROBLEM XII.

TO divide the circumference of a given circle successively into four, eight, twelve, and twenty-four equal parts, by the mere description of circles.

Circular arcs and the radii with which they are described being indicated in corresponding colours, a mere inspection of the diagram, Fig. 23, is all that is necessary for the understanding of the construction. The circle, semicircle, and all the circular arcs traced in *red* are described with the radius A O traced in *red;* the *blue* arcs are described by the double chord, pointed out by the *blue* trace, as radius; the *yellow* are described with the *yellow* straight line as radius; and, lastly, the *black* arcs are described with the *black* straight line as radius. A B = one-fourth the circumference; A C = one-eighth the circumference; B D = one-twelfth the circumference; and C D = one-twenty-fourth of the circumference.

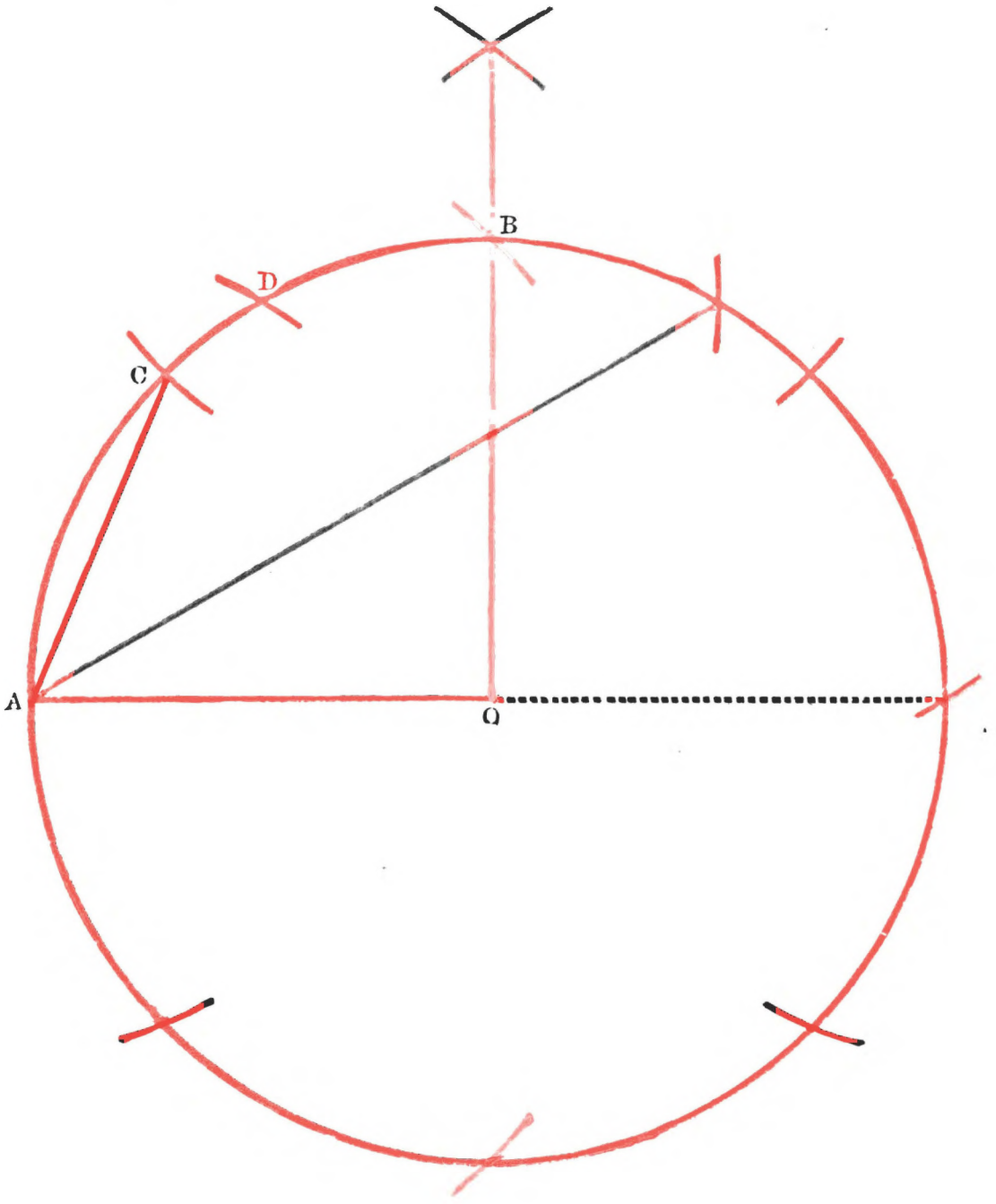

Fig. 23.

DEMONSTRATION OF PROBLEM XII.

MARK out the semicircumference A, D, E, C, by the triple insertion of the radius, from A to C; from the extremities of the diameter A C, with a distance equal to the double chord A E, describe arcs intersecting in the point F; and from A, with the distance B F, cut the circumference at opposite sides at G and H; A G, C G, C H, and H A are quadrants.

$A C^2 = 4 A B^2 = A E^2 + E C^2 \therefore 3 A B^2 = A E^2$. Therefore, $A F^2 = 3 A B^2$; and the triangle A B F being right angled, $3 A B^2 = A B^2 + B F^2$, and hence, $BF^2 = AG^2 = 2AB^2$; and A B G is a right angle, and A G is a quadrant.

From the point F, with the radius A B, cut the circle in I and K, and from A and C lay off the chord A I to L and M; the circumference is divided into eight equal parts by the points A, I, G, K, C, M, H, and L. For $B F^2$ being equal to $2 A B^2$, is equal to the squares of B I and I F. Therefore, B I F is a right angle; but the triangle I B F is also isosceles, and therefore the angle I B F is half a right angle, and the arc I G is the one-eighth part of the circumference. The arc D G, on being repeated, will form twelve equal sections of the circumference. For the arc A D is the sixth or two-twelfths of the circumference, and A G is the fourth or three-twelfths; consequently, the difference D G is one-twelfth. The arc $I D = A D - A I = \frac{1}{6} - \frac{1}{8} = \frac{2}{48} = \frac{1}{24}$ of the circumference.

circumference, angle,

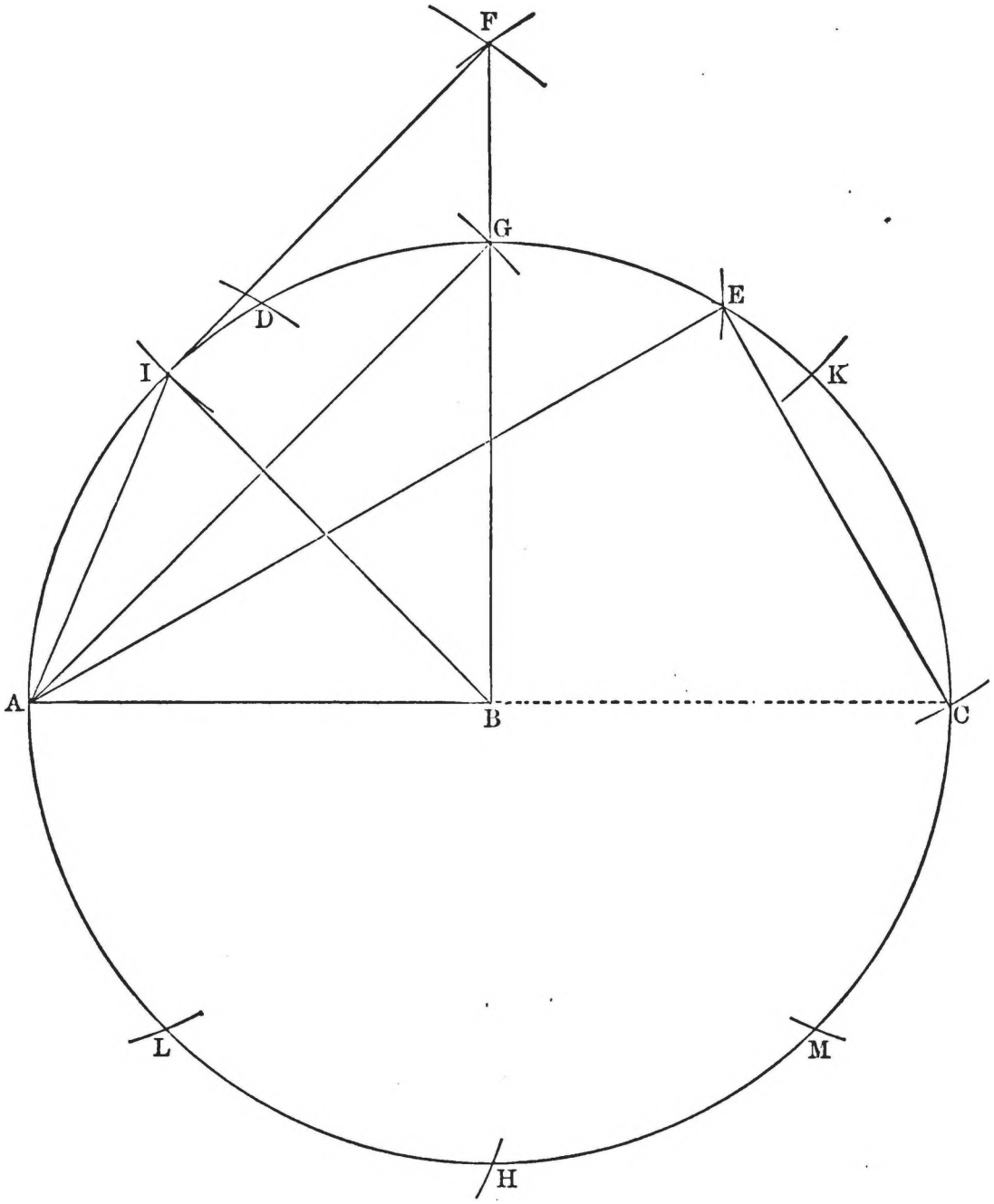

Fig. 24.

PROBLEM XIII.

TO divide the circumference of a given circle successively into five, ten, and twenty equal parts.

Determine, as usual, the semicircumference by the triple insertion of the radius indicated by the *red* trace or straight line. All the arcs traced in *red*, as in the other problems, are described with the radius indicated by the straight line traced in *red*. With the *blue* double chord describe the *blue* arcs; with the *yellow* radius describe the *yellow* arcs; then, the *black* line is the side of the inscribed decagon, and the straight line, indicated by *black* dots, is the side of the inscribed pentagon.

A B $= \frac{1}{4}$ the circumference; A C $= \frac{1}{5}$ the circumference; $\therefore \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$; hence C B is the one-twentieth of the circumference, indicated by the *red* trace; Fig. 25.

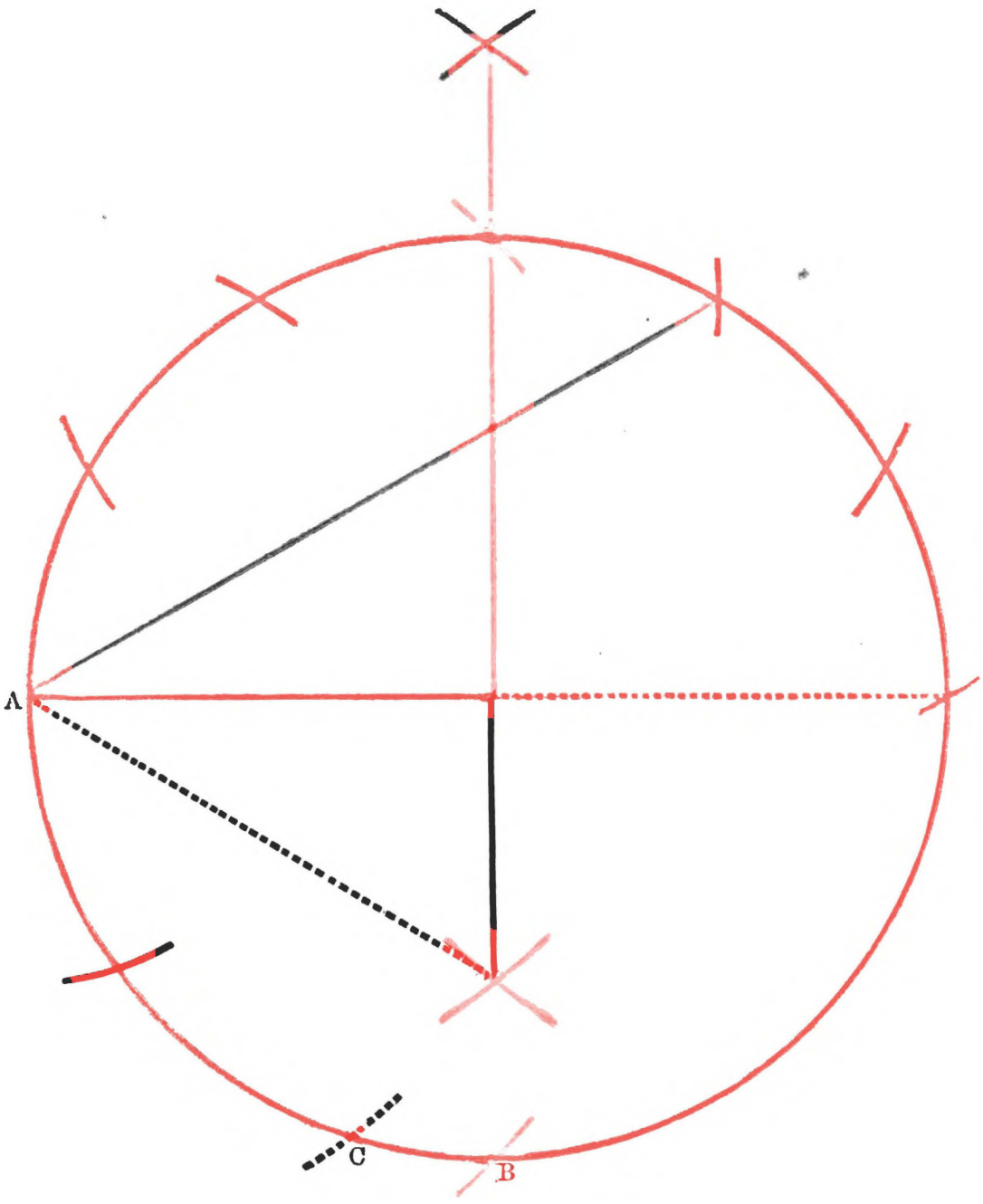

Fig. 25.

DEMONSTRATION OF PROBLEM XIII.

WITH the double chord A E, and centres A, C, describe arcs intersecting in F; from A, with the distance B F, cut the circle in G and K, lay off the chords G H, G I, equal to the radius A B, and, from the points H and I, with the distance B F, describe arcs intersecting in L; Fig. 26.

It is evident, from Problem IX., that B L is the greater segment of the radius B K divided in extreme and mean ratio; therefore, B L = A N is equal to the side of the inscribed decagon; and A L = A M is equal to the side of the inscribed pentagon.

A K = $\frac{1}{4}$ the circumference; A M = $\frac{1}{5}$ the circumference; $\frac{1}{4} - \frac{1}{5} = \frac{1}{20}$. ∴ M K is equal to the twentieth part of the whole circumference A G C K A.

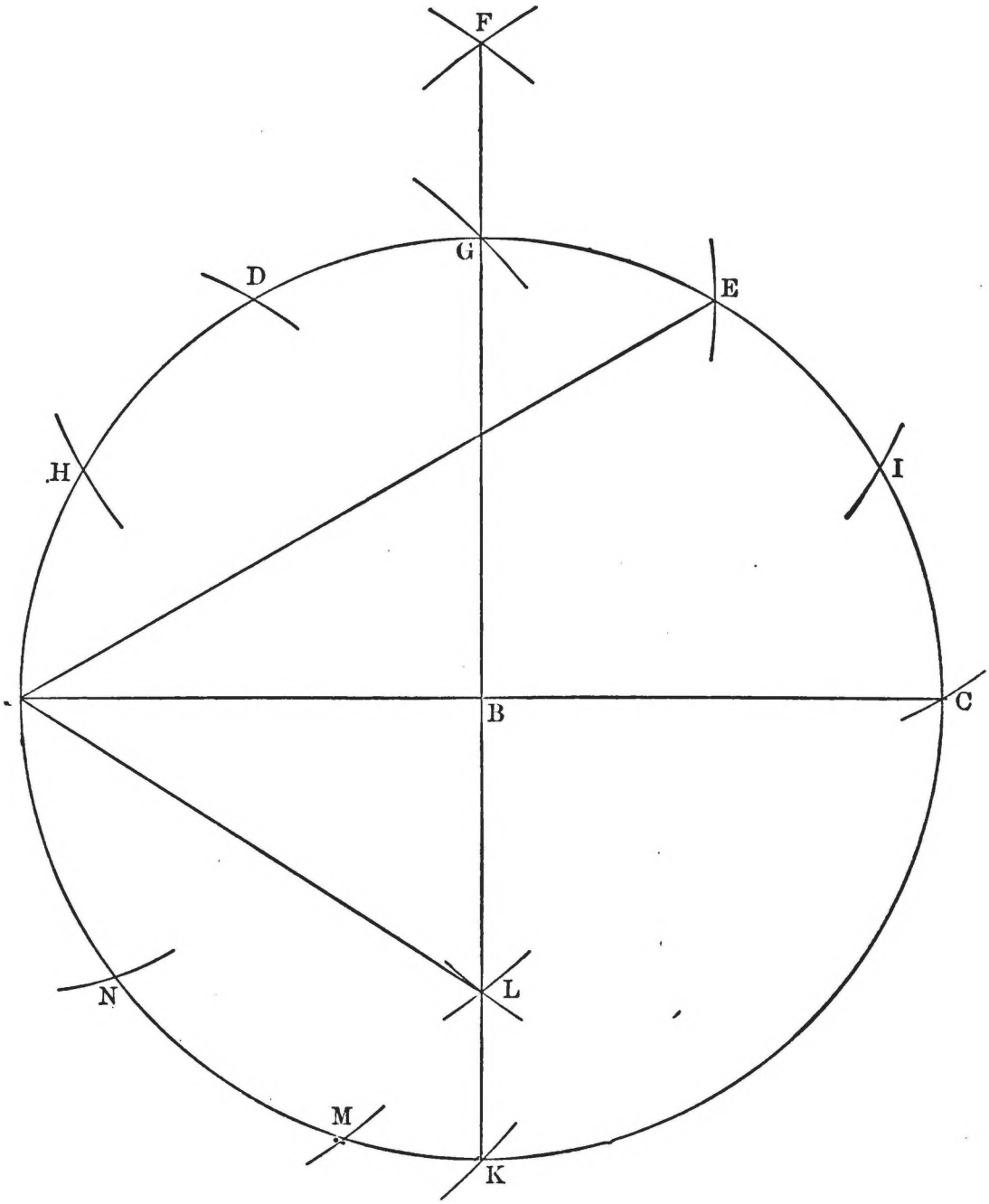

Fig. 26.

PROBLEM XIV.

TO trisect a given distance A B, by compasses only, and no guess work.

Produce the given distance (A B) both ways, until the lines indicated by *black* dots are equal to the *black* line A B; (Problem I.) The arcs traced in *blue* are described by the radii marked *blue*, according to the conventional arrangement. And lastly, the *red* arcs are described with the radius traced in *red;* the two intermediate points I, K, thus determined, are situated at equal intervals in the *black* line, A B; Fig. 27.

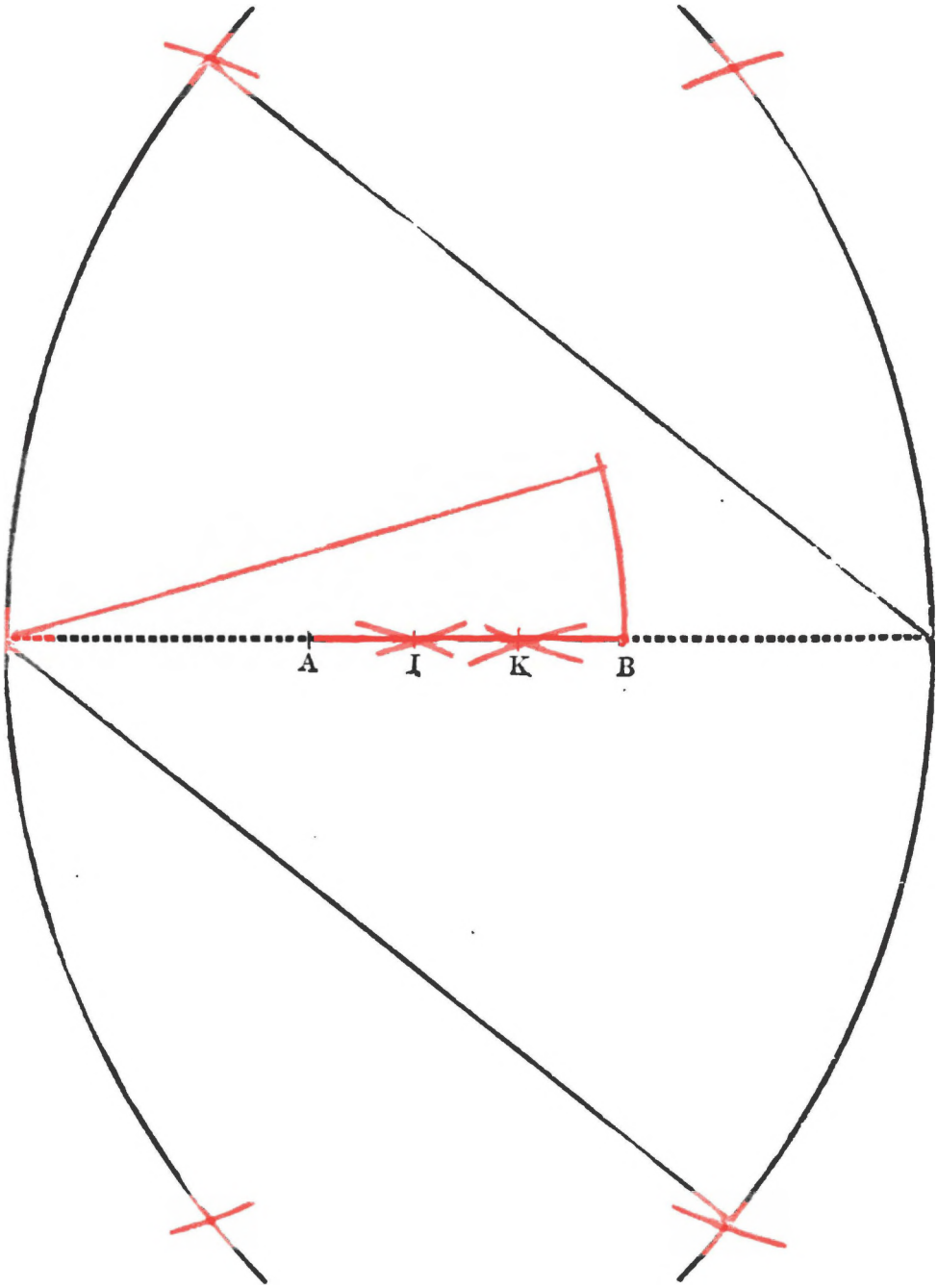

Fig. 27.

DEMONSTRATION OF PROBLEM XIV.

PRODUCE A B, and make A D, and B C, each equal to A B; with the radius C D, describe the arcs E D F and H C G; from D and C lay off chords D E, D F; and C G, C H, all equal to D B or A C; and with the same distance from the four points E, F, G, H, describe arcs intersecting in I and K; the distance A B is thus trisected by points I and K; Fig. 28. It is easily shown, on account of the isosceles triangles, that the points I and K lie in the same direction, A B. Again, K G C being an isosceles triangle, $DG^2 - KG^2 = CD \times DK$; that is, $9AB^2 - 4AB^2$, or $5AB^2 = 3AB \times DK$; $\therefore 5AB = 3DK$; or, $2AB = 3AK$, and $AB = 3BK$. Similar reasoning shows that $AB = 3AI$. Or we may reason thus, let $3x$ represent A B, then $DK = 3x + AK$; and as $5AB = 3DK$, $15x = 9x + 3AK$

$\therefore 6x = 3AK$, or $2x = AK$.

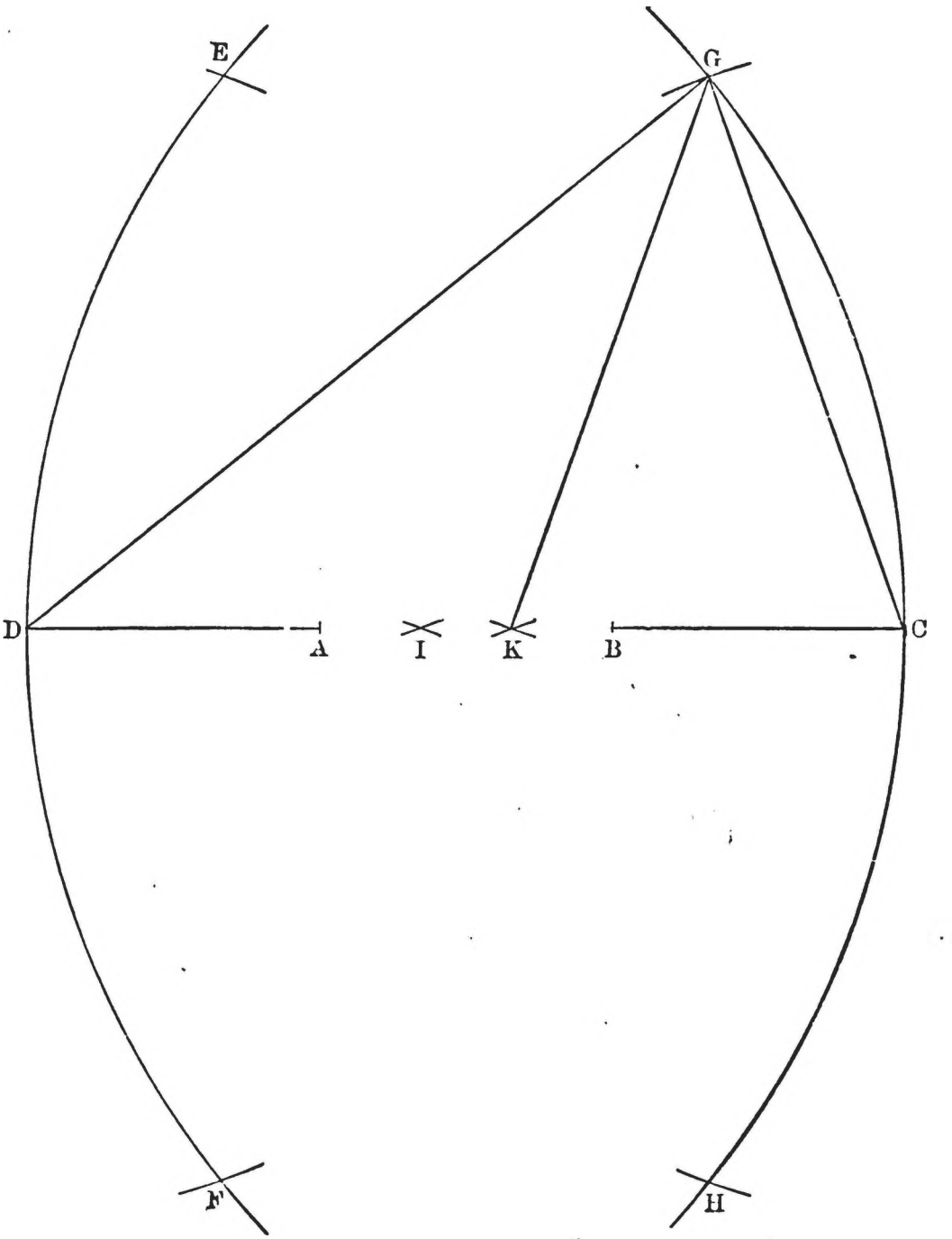

Fig. 28.

PROBLEM XV.

GIVEN two points, A B, to find the intersection of their connecting line with a given circumference.

Describe the *blue* arc with the *blue* radius and the *black* arc with the *black* radius; from the point thus found as centre and the straight line traced in *red*, as radius, describe the *red* arcs; the points in which these arcs cut the given circumference are in the straight line that joins the points A and B; Fig. 29.

From a well-known property of isosceles triangles standing on the same base, the truth of this construction is evident; hence, a demonstration is unnecessary.

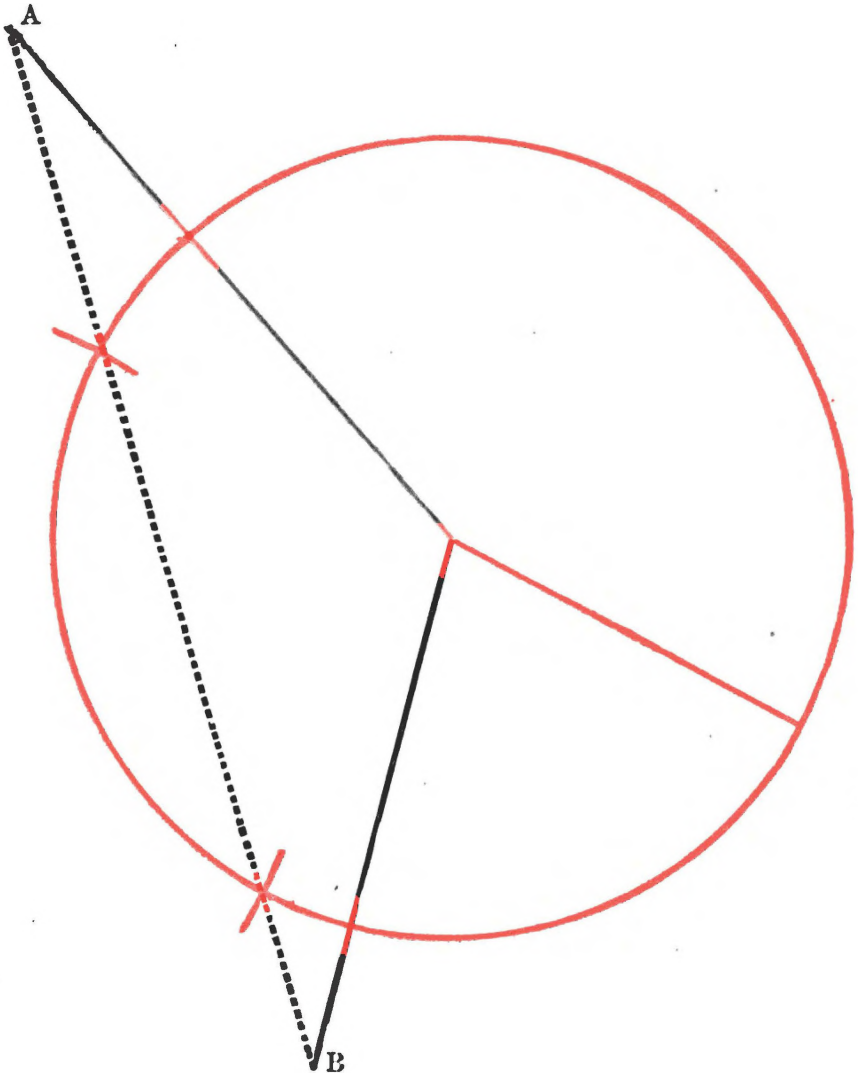

Fig. 29.

PROBLEM XVI.

TO find the sum and difference of two given distances A B, C D, by the description of circles only.

Distances are only indicated by the *red*, *blue*, &c., traces, the straight lines which these traces represent are supposed not to be drawn.

With the *red* line describe the *red* circle; cut the circumference of this circle by an arc like the one indicated in *black;* bisect the arcs into which the *red* circle is thus divided in the points C and D; and B C will be the sum and D B the difference of the two given distances. It requires no words to explain the construction, mere inspection is all that is necessary; see Fig. 30.

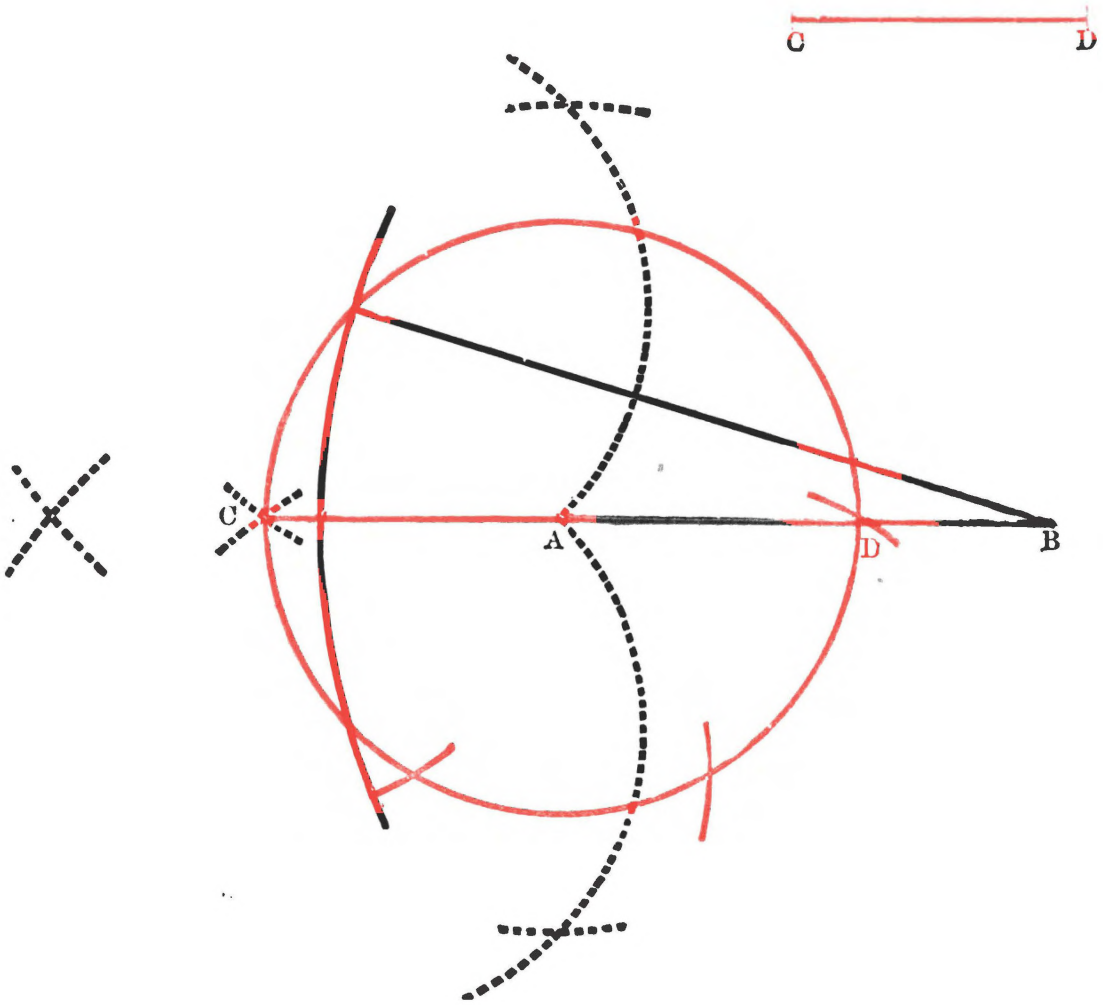

Fig. 30.

E

PROBLEM XVII.

TO find a mean proportional to two given distances, A B, C D.

Let A B and C D be the two distances. To A B add B E equal to C D, by Problem XVI.; bisect A E in F (Problem IV.); and make B G equal to F B (Problem I.).

Describe the semicircumference, and the other arc traced in *red*, with the distance marked *red* as radius; then, the distance indicated by the *blue* trace is the mean proportional required; Fig. 31.

Demonstration. It is evident that B H is perpendicular to A E, and that $B H^2 = A B \times B E$ (Euclid, III. 35); therefore A B : B H :: B H : B E = C D.

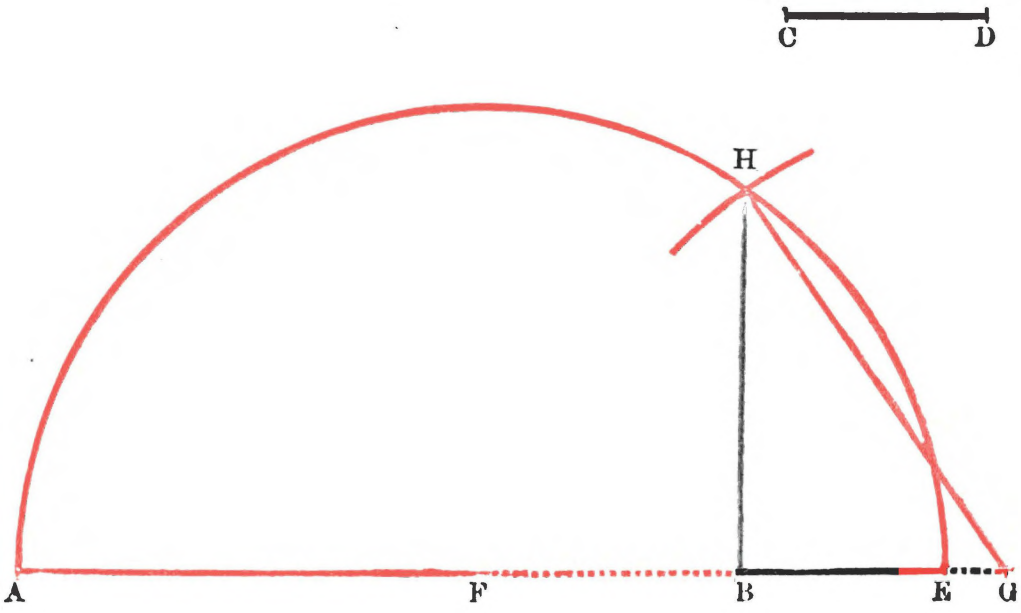

Fig. 31.

PROBLEM XVIII.

TWO distances, A B, C D, being given, to find a third proportional.

From any point describe the circular arc traced in *blue* with the radius or distance traced in *blue;* lay off the *red* chord = the *red* distance; then describe the semicircle and the other circular arcs indicated by *red* traces, with the *red* chord as radius, indicated, as usual, by the corresponding colour; Fig. 32.

The distance pointed out by the chord traced in *black* is the third proportional required.

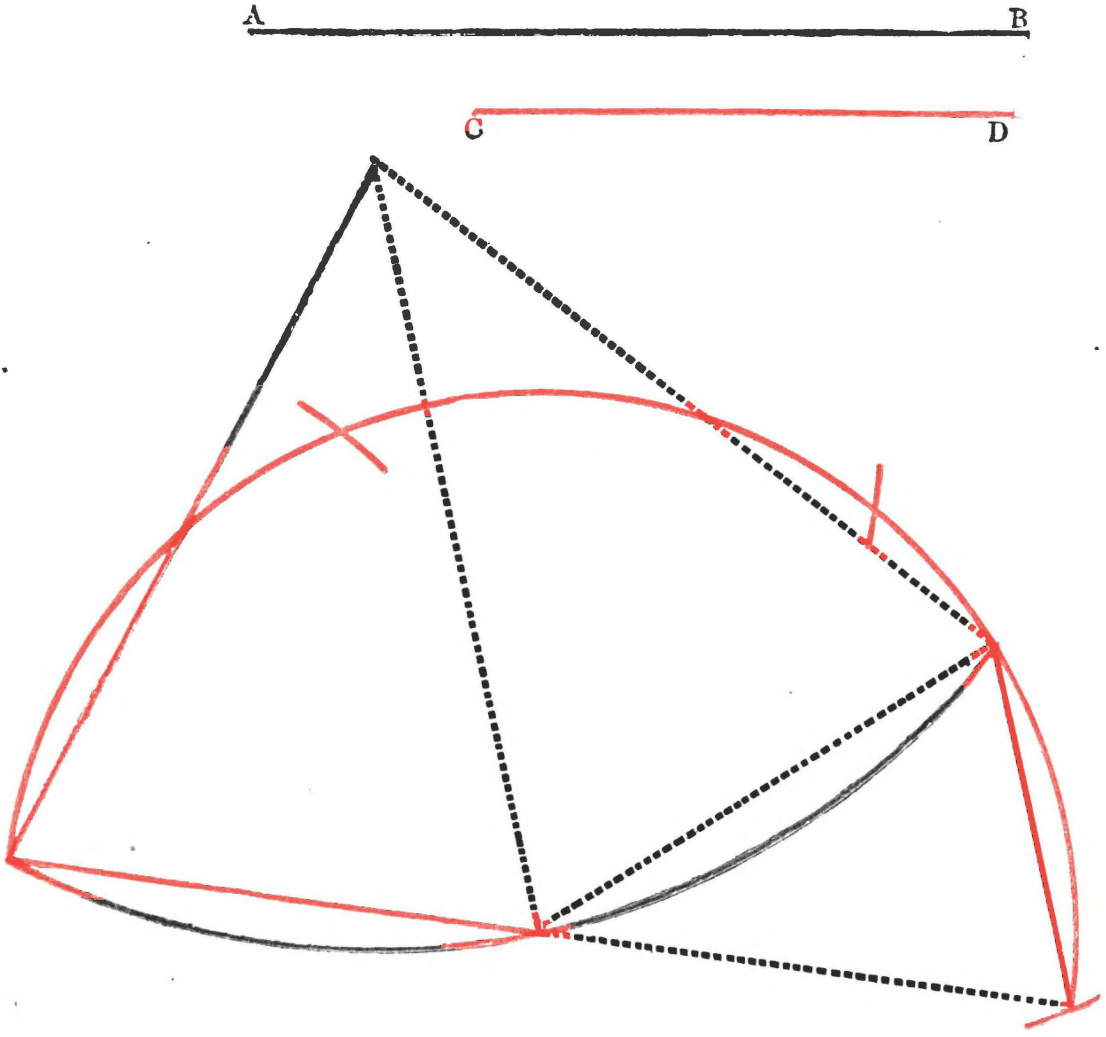

Fig. 32.

DEMONSTRATION OF PROBLEM XVIII.

FROM any point E, and with the distance A B, describe a portion of a circle, in which lay off F G = C D; from G, with G F, describe the semicircle F H I; H I is the third proportional required.

For the angles G E H and I G H, Fig. 33, are each of them double the angle G F H or I F H (Euclid, III. 20); because F G, G H, G I are all equal, and the angle G E H is an angle at the centre of a circle, and H F G is an angle at the circumference, both standing on the same arc G H; therefore, the isosceles triangles G E H and G H I must have all the angles at their bases equal to one another; and therefore the triangles G E H and H G I are similar. Hence (Euclid, VI. 4), E G : G H :: G H : H I; or, A B : C D :: C D : H I.

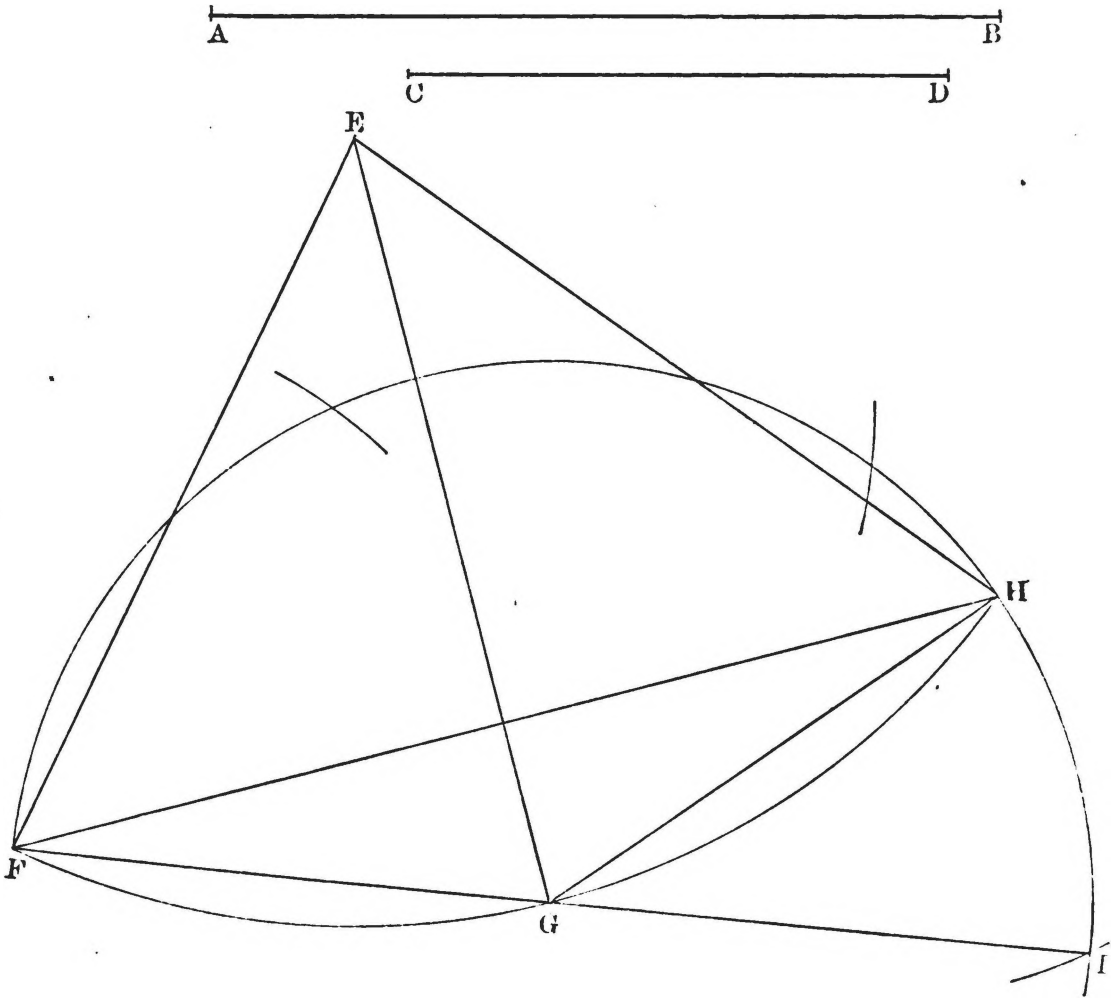

Fig. 33.

PROBLEM XIX.

TO find a fourth proportional to three given distances, A B, C D, E F, by the use of the compasses only.

Describe the *yellow* circle with the *yellow* straight line; the *red* circle with the *red* straight distance; assume any point K in the circumference traced in *red*, and describe the *blue* arc with the radius marked in *blue;* then the distance represented by the chord dotted *black* is a fourth proportional to the distances A B, C D, and E F; Fig. 34.

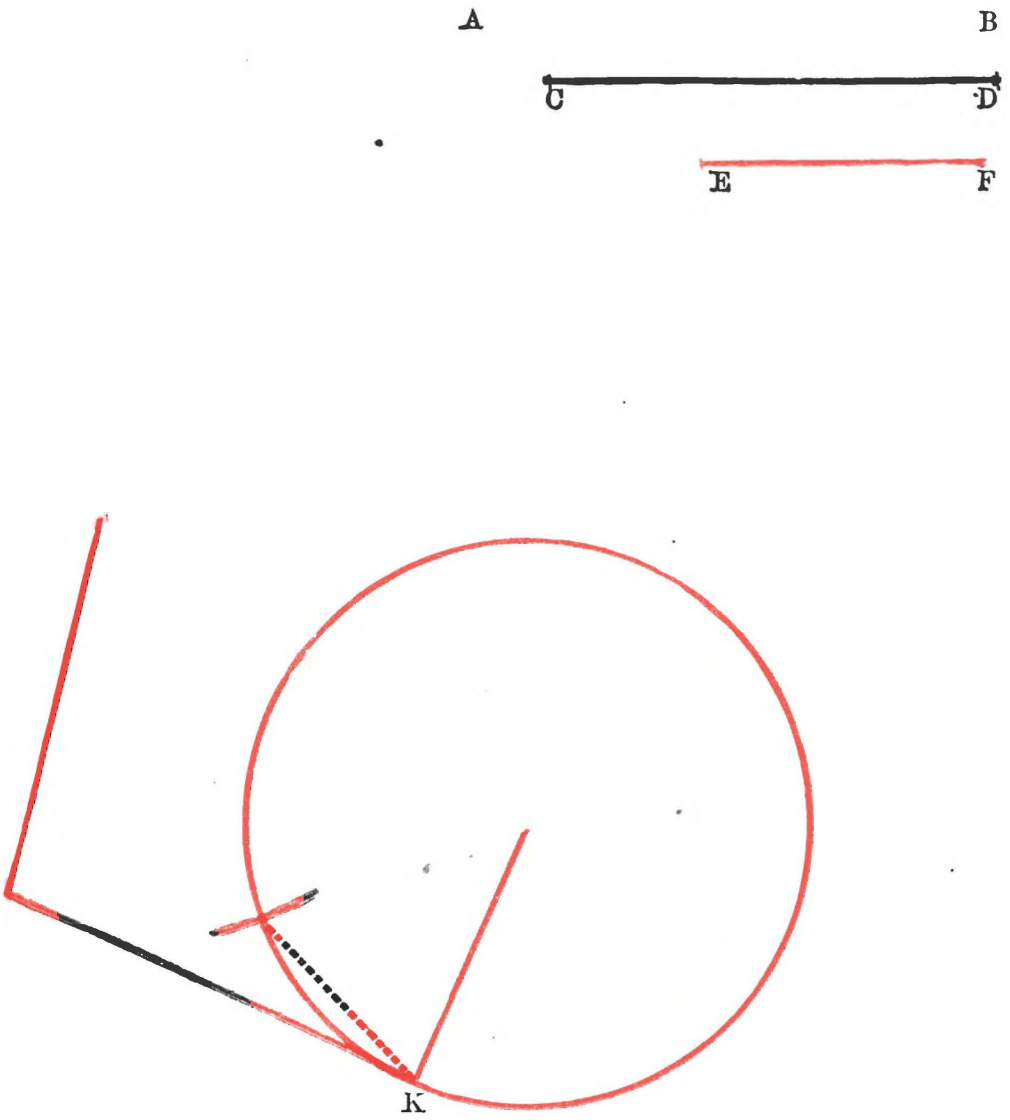

Fig. 34.

DEMONSTRATION OF PROBLEM XIX.

FROM any point G, describe two concentric circles with I G = A B, and G K = E F. In the first circle lay off H I = C D; assume any point K in the second circumference, which cut in L by an arc described from I with the distance H K; the chord L K is the fourth proportional required; Fig. 35. For the triangles I L G and H K G are equal, since their corresponding sides are evidently equal; whence the angle I G L is equal to H G K; take away the angle H G L, then the angle I G H remains equal to L G K. Therefore, the isosceles triangles G H I and G L K are similar (Euclid, VI. 4); and therefore, G I : I H :: G L : L K, that is, A B : C D :: E F : L K.

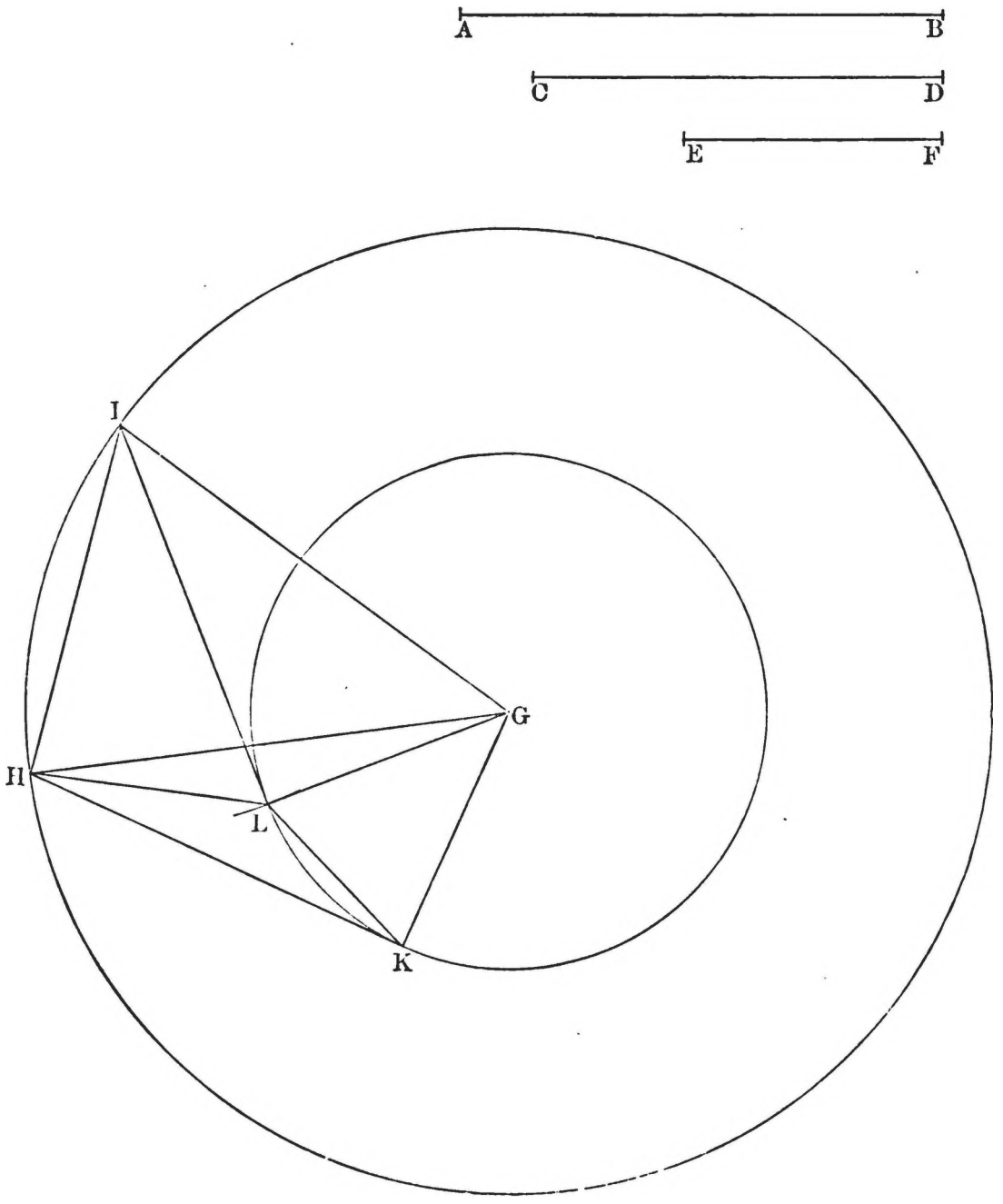

Fig. 35.

PROBLEM XX.[1]

TO divide the circumference of a given circle, B T A Z, into seven equal parts by plane geometry.

The following construction may be employed when the arc D P, or the arc P *n* Q D of the circle D Q *n* P is trisected; Fig. 36. Take O A = A D = D E = 1; on O D describe the equilateral triangle O F D and draw F E. Draw D H parallel to O F, and E H will be the radius of the circle D P *n* Q; Fig. 36. Take D P = the chord, = $\frac{1}{3}$ of D E, or = $\frac{1}{3}$ of the radius O A, then the straight D G = $\frac{1}{2}$ the straight line D P = $\frac{1}{6}$th of O A; C G = $\frac{1}{2}$ A F. In the right-angled triangle C D G, C D = H E = $\frac{\sqrt{7}}{3}$ = the radius of the circle D P *n* Q; D G = G P = $\frac{1}{6}$ of A O; and C G = $\frac{1}{2}$ A F = $\frac{\sqrt{3}}{2}$. The tangent of the angle C D G = $\frac{\text{C G}}{\text{D G}} = \frac{\sqrt{3}}{2}$ divided by $\frac{1}{6}$ = $3\sqrt{3}$. Let P *b* be equal to one-third of the arc P *b a* D; then, if *b n* be the side of an equilateral triangle inscribed in the circle, P *n* will be one-third of the greater arc P *n* Q D; Q *n*, *n b*, are the sides of an inscribed equilateral triangle. Make P L = P Q, and draw L R perpendicular to O A meeting the semicircle O R A in R. Then O R is equal to the side of the regular heptagon B S T V X Y Z.

[1] This problem, proposed by a profound mathematician, Matthew Collins of Dublin, was first solved and demonstrated by Oliver Byrne in his work, "Dual Arithmetic, a New Art," part II., pp. 179, 180.

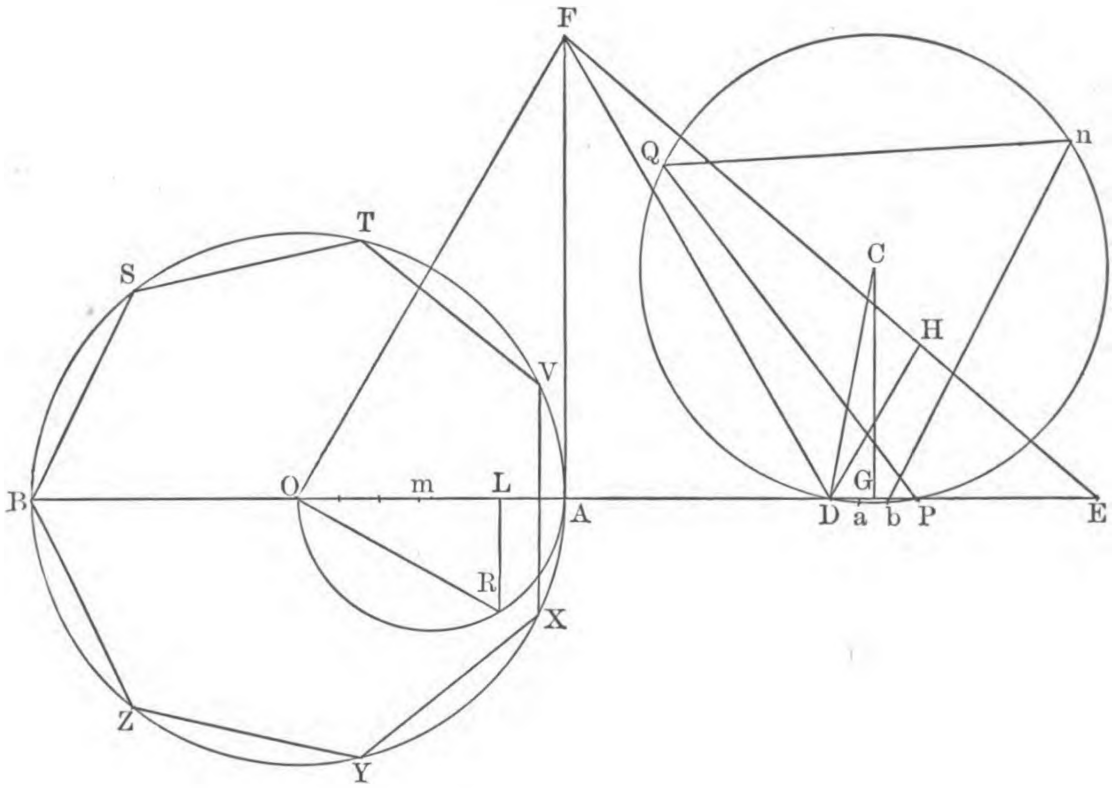

Fig. 36.

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